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This note firstly introduces the basic option trading strategies and the “Greek letters” of the Black-Scholes model. It further discusses various market quoting conventions for the at-the-money and delta styles, and then summarizes the definition of the market quoted at-the-money, risk reversal and strangle volatilities. A volatility surface can be constructed from these volatilities which provides a way to interpolate an implied volatility at any strike and maturity from the surface. At last, the vanna-volga pricing method [1] is presented which is often used for pricing first-generation FX exotic products. An simple application of the method is to build a volatility smile that is consistent with the market quoted volatilities and allows us to derive implied volatility at any strike, in particular for those outside the basic range set by the market quotes.

1. Trading Strategies of Vanilla Options

In the following, we will introduce a few simple trading strategies based on vanilla options. These products are traded liquidly in FX markets.

1.1. Single Call and Put

The figures below depict the payoff functions of vanilla options.

1.2. Call Spread and Put Spread
A call spread is a combination of a long call and a short call option with different strikes \( K_1 < K_2 \).

A put spread is a combination of a long put and a short put option with different strikes. The figure below shows the payoff functions of a call spread and a put spread.

\[
\text{CallSpread} = C(K_1) - C(K_2) \\
\text{PutSpread} = P(K_2) - P(K_1)
\]

1.3. Risk Reversal

A risk reversal (RR) is a combination of a long call and a short put with different strikes \( K_1 < K_2 \).

This is a zero-cost product as one can finance a call option by short selling a put option. The figure below shows the payoff function of a risk reversal.

\[
\text{RiskReversal} = C(K_1) - P(K_2)
\]

1.4. Straddle and Strangle

A straddle is a combination of a call and a put option with the same strike \( K \). A strangle is a combination of an out-of-money call and an out-of-money put option with two different strikes \( K_1 < K_{ATM} < K_2 \). The figure below shows the payoff functions of a straddle and a strangle.

\[
\text{Straddle} = C(K) + P(K) \\
\text{Strangle} = C(K_1) + P(K_2)
\]

1.5. Butterfly
A butterfly (BF) is combinations of a long strangle and a short straddle. The figure below shows the payoff function of a butterfly

\[ \text{Butterfly} = \text{Strangle}(K_1, K_2) - \text{Straddle}(K) \]

\[ K_1 < K, K_{\text{ATM}} < K_2 \]

2. FX OPTION QUOTING CONVENTION

2.1. Black-Scholes Formula

Currency pairs are commonly quoted using ISO codes in the format FORDOM, where FOR and DOM denote foreign and domestic currency respectively. For example in EURUSD, the EUR denotes the foreign currency or currency1 and USD the domestic currency or currency2. The rate of EURUSD tells the price of one euro in USD.

In Black-Scholes model, FX spot rate is assumed to follow a geometric Brownian motion. Under domestic risk neutral measure, the FX spot is characterized by the following stochastic differential equation with a drift \( r - \hat{r} \) and a volatility \( \sigma \)

\[
\frac{dS_t}{S_t} = (r - \hat{r})dt + \sigma d\hat{W}_t
\]  (1)

where the \( r \) and \( \hat{r} \) are the domestic and foreign risk free rate respectively (the accent hat “^” here denotes a counterpart in the foreign economy, e.g. \( \hat{r} \) is the foreign risk free rate). With the assumption of deterministic interest rates, an option on the FX spot with a strike \( K \) can be priced in Black-Scholes model

\[
V_t = \mathcal{B}(\omega, K, \sigma, \tau) = \omega \hat{P}S\Phi(\omega d_+) - \omega P\hat{K}\Phi(\omega d_-) = \omega P\left(F\Phi(\omega d_+) - K\Phi(\omega d_-)\right)
\]  (2)

where we define \( \omega = 1 \) or \(-1\) for call or put, \( \tau = T - t \) for term to maturity, \( \Phi \) for standard normal cumulative density function, and \( d_+ \) and \( d_- \) as follows
\[ d_\pm = \frac{1}{\sigma \sqrt{t}} \ln \frac{F}{K} - \frac{\sigma \sqrt{t}}{2} \quad \text{and} \quad d_+ = d_- + \sigma \sqrt{t} \]  

(3)

In (2), the \( P_{t,T} \) and \( \hat{P}_{t,T} \) denote the domestic and the foreign zero coupon bond price (or equivalently the discount factors if rates are deterministic), respectively. The FX forward \( F \) is given by the **covered interest rate parity** (i.e. the returns from investing domestically must be equal to the returns from investing abroad to be arbitrage-free)

\[
F_{t,T} = S_t \frac{\hat{P}_{t,T}}{P_{t,T}}, \quad P_{t,T} = \exp \left( -\int_t^T r_u du \right), \quad \hat{P}_{t,T} = \exp \left( -\int_t^T \hat{r}_u du \right) \tag{4}
\]

FX options are usually physically settled (i.e., upon exercise at maturity, the buyer of a EURUSD call receives notional \( N \) amount in EUR and pays \( NK \) amount in USD).

**Black-Scholes pricing formula** can be easily derived from arbitrage-free pricing

\[
V_t = \mathbb{E}_t \left[ \frac{M_t}{M_T} (S_T - K)^+ \right] = \mathbb{E}_t \left[ \frac{M_t}{M_T} S_T \mathbb{1}\{S_T > K\} \right] - K \mathbb{E}_t \left[ \frac{M_t}{M_T} \mathbb{1}\{S_T > K\} \right] \\
= \mathbb{E}_t^S \left[ \frac{\hat{M}_t S_T}{\hat{M}_T S_T} \mathbb{1}\{S_T > K\} \right] - K \mathbb{E}_t^T \left[ \frac{P_{t,T}}{P_{T,T}} \mathbb{1}\{S_T > K\} \right], \quad \text{Change } M_t \rightarrow \hat{M}_t S_t, \; M_t \rightarrow P_{t,T} \\
= S_t \mathbb{E}_t^S \left[ \frac{\hat{M}_t}{\hat{M}_T} \mathbb{1}\{S_T > K\} \right] - K \mathbb{E}_t^T \left[ \frac{P_{t,T}}{P_{T,T}} \mathbb{1}\{S_T > K\} \right] \\
= S_t \hat{P}_{t,T} \mathbb{P}_t^S[S_T > K] - K P_{t,T} \mathbb{P}_t^T[S_T > K] \quad \text{by deterministic rates} \\
= \hat{P} S \Phi(d_+) - PK \Phi(d_-)
\]

where \( \mathbb{P}_t^S[S_T > K] \) and \( \mathbb{P}_t^T[S_T > K] \) are both conditional probabilities of spot finishing in-the-money at maturity. The \( \mathbb{P}_t^S[S_T > K] \) is computed under the measure associated with the foreign money market account denominated in domestic currency \( \hat{M}_t S_t \) as the numeraire, whereas the \( \mathbb{P}_t^T[S_T > K] \) is computed under \( T \)-forward measure associated with domestic zero coupon bond \( P_{t,T} \) as the numeraire. Since the drift adjustment due to change of numeraire is

\[
d\hat{W}_t = d\hat{W}_t^N + \sigma_N dt \tag{6}
\]
where $\mathbb{N}$ denotes the measure associated with numeraire $N$ and $\mathbb{Q}$ the risk neutral measure. The FX spot process under the measure with itself (times $\mathbb{M}_t$) as the numeraire is given by

$$
\frac{dS_t}{S_t} = (r - \hat{r})dt + \sigma d\hat{W}_t = (r - \hat{r} + \sigma^2)dt + \sigma dW_t^S
$$

The total drift adjustment $\sigma^2 \tau$ for period $\tau = T - t$ is then normalized by the total volatility $\sigma \sqrt{\tau}$ of the stock to give a shift term $\sigma \sqrt{\tau}$ as the difference between $d_+$ and $d_-$ in the classic Black-Scholes formula.

2.2. Foreign-Domestic Symmetry

On top of the well-known put-call parity in options, there exists a foreign-domestic symmetry in currency options, shown as below

$$
\frac{1}{S} \cdot \text{OptionValue}(\omega, S, K, \sigma, r, \hat{r}, \tau) = K \cdot \text{OptionValue}\left(-\omega, \frac{1}{S}, \frac{1}{K}, \sigma, \hat{r}, r, \tau\right)
$$

For example, a call on $S$ is equivalent to a put on $\hat{S}$. Alternatively speaking, a right to buy one FOR at a price of $K$ DOM is equivalent to the right to sell $K$ DOM at a price of one FOR. In Black-Scholes model, the symmetry can be derived as follows, e.g. the value of a call on $S$ is

$$
V = P_{t,T} \left(F\Phi(d_+) - K\Phi(d_-)\right), \quad d_+ = \frac{1}{\sigma \sqrt{\tau}} \ln \frac{F}{K} + \frac{\sigma \sqrt{\tau}}{2}, \quad d_- = d_+ - \sigma \sqrt{\tau}
$$

and the value of a put on $\hat{S}$ is

$$
\hat{V} = -\hat{P}_{t,T} \left(F\Phi(-d_+) - K\Phi(-d_-)\right) = \hat{P}_{t,T} \frac{F\Phi(d_+) - K\Phi(d_-)}{FK} = \frac{V}{SK}
$$

and

$$
\hat{d}_+ = \frac{1}{\sigma \sqrt{\tau}} \ln \frac{\hat{F}}{\hat{K}} + \frac{\sigma \sqrt{\tau}}{2} = -d_-, \quad \hat{d}_- = \hat{d}_+ + \sigma \sqrt{\tau} = -d_+
$$

2.3. Market Quoting Convention

The option price quoting convention varies for currencies [2] [3]. Options can be quoted in one of the four relative quote styles: domestic per foreign ($fd$), percentage foreign ($%f$), percentage domestic ($%d$) and foreign per domestic ($df$). The call and put prices we showed in defined in (2) are actually expressed in domestic per foreign style (also known as the domestic pips price), denoted by $V_{fd}$. With the
notional amount $N$ expressed in foreign currency, we have $V_{fd} = N \mathbb{B}(\omega, K, \sigma, \tau)$. The other price quote styles have the following relationships with respect to $V_{fd}$

$$V_{\%f} = \frac{V_{fd}}{S}, \quad V_{\%d} = \frac{V_{fd}}{K}, \quad V_{df} = \frac{V_{fd}}{SK} \quad (11)$$

It is very important to note that this technique of constructing all these different quote styles only works where there are two notional s given by strike $K = N/\tilde{N}$, in foreign and domestic currencies, and there is a fixed relationship between them, which is known from the start. This is true for European and American style vanilla options, even in the presence of barriers and accrual features, but is most definitely not true for digital options. Suppose one has a cash-or-nothing digital which pays one USD if the EURUSD FX rate fixes at time $T$ above a particular level (sometimes called ‘strike’, which actually leads to the confusion). The digital clearly has a USD notional (= $1, the domestic notional) so we can obtain percentage domestic (%USD) and foreign per domestic (EUR/USD) prices. However, there is no EUR notional (the foreign notional) at all so the other two quote styles are meaningless [4].

3. **RISK SENSITIVITIES**

Risk sensitivity of an option is the sensitivity of the price to a change in underlying state variables or model parameters. We will present some basic types of risk sensitivities in the context of Black-Scholes model.

3.1. **Delta**

Delta is the ratio of change in option value to the change in spot or forward. There are several definitions of delta, such as spot/forward delta, pips/percentage delta, etc. Since FX volatility smiles are commonly quoted as a function of delta rather than as a function of strike, it is important to use a delta definition consistent with the market convention for the currency.

3.1.1. **Pips Spot Delta**

The pips spot delta is defined in Black-Scholes model as the first derivative of the present value with respect to the spot, both in domestic per foreign terms, corresponding to risk exposures in FOR. This
style of delta implies that the premium currency is DOM and notional currency is FOR. It is commonly adopted by currency pairs with USD as DOM (or currency2), e.g. EURUSD, GBPUSD and AUDUSD, etc. By assuming \( N = 1 \) in FOR and hence \( V_{fd} = \mathcal{B}(\omega, K, \sigma, \tau) \), the pips spot delta is equivalent to the standard Black-Scholes delta

\[
\Delta = \frac{\partial V_{fd}}{\partial S} = \omega \hat{P} \Phi(\omega d_+) + \omega \hat{P} S \phi(d_+ \omega) \frac{\partial d_+}{\partial S} - \omega PK \phi(d_- \omega) \frac{\partial d_-}{\partial S} = \omega \hat{P} \Phi(\omega d_+) 
\]

(12)

where we have used the following identities

\[
\frac{\partial \Phi(\omega d_+)}{\partial d_+} = \omega \phi(d_+) = \frac{\omega}{\sqrt{2\pi}} \exp \left( \frac{-d_+^2}{2} \right), \quad \frac{\partial \Phi(\omega d_-)}{\partial d_-} = \omega \phi(d_-) = \frac{\omega}{K} F \phi(d_+) 
\]

(13)

\[
\Phi(-x) = 1 - \Phi(x), \quad \frac{\partial d_+}{\partial S} = \frac{\partial d_-}{\partial S} = \frac{1}{S \sigma \sqrt{\tau}} 
\]

with \( \phi \) the normal probability density function. To understand pips spot delta, assuming DOM is the numeraire, if one wants to hedge a short call of \( N \) notional in FOR with a premium of \( NV_{fd} \) in DOM, he must be long \( N\Delta \) amount of the spot \( S \). This can be achieved by entering a long position of \( N\Delta \) units of FOR with a cost of \( N\Delta S \) units of DOM.

3.1.2. Percentage Spot Delta

The percentage spot delta (also known as \textit{premium adjusted} pips spot delta) is defined as a derivative of the present value with respect to the spot, both in percentage foreign terms, corresponding to risk exposures in DOM. This style of delta implies that the premium currency and notional currency both are FOR. It is used by currency pairs like USDJPY, EURGBP, etc. In Black-Scholes model, the percentage spot delta has the form

\[
\Delta_\% = \frac{\partial V_{\%f}}{\partial S} = S \frac{\partial}{\partial S} \left( \frac{V_{fd}}{S} \right) = \frac{\partial V_{fd}}{\partial S} - \frac{V_{fd}}{S} = \Delta - V_{\%f} = \omega \hat{P} \frac{K}{F} \Phi(\omega d_-) 
\]

(14)

which shows that the percentage spot delta is the pips spot delta \textit{premium-adjusted} by percentage foreign option value. This can be explained by assuming FOR is the numeraire. If one wants to hedge a short call
of $N$ notional in FOR with a premium of $NV_{fd}/S$ in FOR, the delta sensitivity with respect to the spot inverse $\hat{S} \equiv 1/S$ must be

$$\frac{\partial V_{fd}}{\partial \frac{1}{S}} = \frac{1}{S} \frac{\partial V_{fd}}{\partial S} \frac{\partial S}{\partial S} - \frac{1}{S^2} \frac{\partial S}{\partial S} = V_{fd} - S\Delta$$

(15)

To hedge the delta risk, one must be long $N(V_{fd} - S\Delta)$ amount of the spot inverse $1/S$. This can be achieved by entering a long position in $N(V_{fd} - S\Delta)$ units of DOM with a cost of $N(V_{fd}/S - \Delta)$ units of FOR. Or equivalently, one enters a long position in $N(\Delta - V_{fd}/S)$ units of FOR with a cost of $N(S\Delta - V_{fd})$ units of DOM, which translates exactly into the percentage spot delta $\Delta_{\%} = \Delta - V_{\%f}$.

Whether pips or percentage delta is quoted in markets depends on which currency in the currency pair FOR/DOM is the premium currency, and the definition of premium currency itself is a market convention. If the premium currency is DOM, then no premium adjustment is applied and the pips delta is used, whereas if the premium currency is FOR then the percentage delta is used. Despite the fact that market convention involves different delta quotation styles, they are mutually equivalent to one another (referring to [5] for more details). The difference between pips delta and percentage delta comes naturally from the change of measure between domestic and foreign risk-neutral measures. Consider the case of a call option on FOR/DOM, or to be more thorough, a FOR call/DOM put. If the two counterparties to such a trade are FOR based and DOM based respectively, then they will agree on the price. However, the price will be expressed and actually exchanged in one of two currencies: FOR or DOM. From a domestic investor’s point of view, if the premium currency is DOM, the premium itself is riskless and the hedging of the option can be done by simply taking $\Delta$ amount of FOR/DOM spot. If however the premium currency is FOR, there will be two sources of currency risk: 1) the change in intrinsic option value due to the move in underlying spot. 2) the change in premium value converted from FOR to DOM due to the move in FX rate. Apparently to hedge the two risks, one must take $\Delta_S$ and $-V_{fd}/S_t$ amount of spot position respectively. Alternatively speaking, the premium adjustment comes from the fact that a premium in FOR
would have already hedged part of the option’s delta risk [6], which must be accounted in calculating the delta.

3.1.3. Pips Forward Delta

The pips forward delta is the ratio of the change in forward value (in contrast to present value!) of the option to the change in the relevant FX forward, both in domestic per foreign terms

$$
\Delta_F = \frac{\partial V_{F,fd}}{\partial F} = \omega \Phi(\omega d_+) = \frac{\Delta}{\tilde{P}}
$$

by the following facts

$$
V_{F,fd} = \frac{V_{fd}}{P} = \omega F \Phi(\omega d_+) - \omega K \Phi(\omega d_-), \quad \frac{\partial d_+}{\partial F} = \frac{\partial d_-}{\partial F} = \frac{1}{F \sigma \sqrt{t}}
$$

3.1.4. Percentage Forward Delta

The percentage forward delta is defined as the ratio of the change in forward value of the option to the change in the FX forward, both in percentage foreign terms

$$
\Delta_{%F} = \frac{\partial V_{F,\%f}}{\partial F} = F \frac{\partial}{\partial F} \left( \frac{V_{F,fd}}{F} \right) = \frac{\partial V_{F,fd}}{\partial F} - \frac{V_{F,fd}}{F} = \Delta_F - \frac{V_{F,\%f}}{F} = \omega \frac{K}{F} \Phi(\omega d_-)
$$

Again, the percentage forward delta is the pips forward delta premium-adjusted by forward percentage foreign option value.

The choice between spot delta and forward delta depends on the currency pair as well as the option maturity. Spot delta is mainly used for tenors less than or equal to 1Y and for the currency pair with both currencies from the more developed economies. Otherwise, the use of forward delta dominates. It is obvious that the spot delta and forward delta differ only by a foreign discount factor $\tilde{P}_{t,T}$. Since the credit crunch of 2008 and the associated low levels of liquidity in short-term interest rate products, it became unfeasible for banks to agree on spot deltas (which include discount factors) and, as a result, market practice has largely shifted to using forward deltas exclusively in the construction of the FX smile, which do not include any discounting [7].

3.1.5. Strike from Delta Conversion
It is straightforward to compute strikes from pips deltas. However, since explicit strike expressions in percentage deltas are not available, we must solve for the strikes numerically. It can be seen that the percentage deltas are monotonic in strike on put side, but this is not the case on call side. Using percentage forward delta as an example, the expression of a call delta is

$$\Delta_{\%F} = \frac{K}{F} \Phi\left(\frac{1}{\sigma \sqrt{\tau}} \ln \frac{F}{K} - \frac{\sigma \sqrt{\tau}}{2}\right)$$

(19)

Obviously, the delta has two sources of dependence on strike and the function is not always monotonic. This may result in two different solutions of strike. To avoid the undesired solution, the numerical search can be performed within a range \((K_{\text{min}}, K_{\text{max}})\) that encloses the proper strike solution. We can choose the strike by pips delta as the upper bound \(K_{\text{max}}\) (because a pips delta maps to a strike that is always larger than that of a percentage delta) and the lower bound \(K_{\text{min}}\) can be found numerically as a solution to the equation below (where \(K_{\text{min}}\) maximizes the \(\Delta_{\%F}\)) \[8\]

$$\frac{\partial \Delta_{\%F}}{\partial K} = \frac{\Phi(d_-)}{F} - \frac{1}{F \sigma \sqrt{\tau}} \phi(d_-) = 0 \Rightarrow \Phi(d_-) \sigma \sqrt{\tau} = \Phi(d_-)$$

(20)

However, the function below

$$f(K) = \Phi(d_-) \sigma \sqrt{\tau} - \Phi(d_-)$$

(21)

is also not monotonic. It has a maximum \(\sigma \sqrt{\tau}\) when \(K \to 0\) and a minimum when \(K = F \exp\left(\frac{1}{2} \sigma^2 \tau\right)\), which can be used to find the \(K_{\text{min}}\). The table below summarizes the delta and strike conversion of the 4 delta conventions.

<table>
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<th>Delta Convention</th>
<th>Delta from Strike</th>
<th>Strike from Delta</th>
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<tbody>
<tr>
<td>pips spot</td>
<td>$$\Delta(K) = \omega \hat{P} \Phi(\omega d_+)$$</td>
<td>$$K(\delta</td>
</tr>
<tr>
<td>pips forward</td>
<td>$$\Delta_F(K) = \omega \Phi(\omega d_+)$$</td>
<td>$$K(\delta</td>
</tr>
<tr>
<td>percentage spot</td>
<td>$$\Delta_{%}(K) = \omega \hat{P} \frac{K}{F} \Phi(\omega d_-)$$</td>
<td>$$K(\delta</td>
</tr>
</tbody>
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Changwei Xiong, January 2021

percentage forward \( \frac{\Delta_{\%}}{F} (K) = \omega \frac{K}{F} \Phi(\omega d_{-}) \in (K_{min}, K(\delta|\Delta_{\%})) \) for \( \omega = 1 \)

3.2. Other Risk Sensitivities

In the following context, we will only express the risk sensitivities in domestic per foreign terms for simplicity. Assuming the value of option is given in the Black-Scholes model, e.g. \( V_{f,d} = \mathcal{B}(\omega, K, \sigma, \tau) \), the risk sensitivities can be derived as follows.

3.2.1. Theta

Theta \( \theta \) is the first derivative of the option price with respect to the initial time \( t \). Converting from \( t \) to \( \tau \), we have \( \theta = \partial \mathcal{B} / \partial t = -\partial \mathcal{B} / \partial \tau \). Let’s first derive the partial derivatives

\[
\begin{align*}
\frac{\partial d_{+}}{\partial \tau} &= \frac{\partial \left( \left( \frac{\mu + \sigma}{2} \right) \sqrt{\tau} + \frac{1}{\sigma \sqrt{\tau}} \ln \frac{S_{t}}{K} \right)}{\partial \tau} = \frac{\mu}{2\sigma \sqrt{\tau}} + \frac{\sigma}{4\sqrt{\tau}} - \frac{1}{2\sigma \sqrt{\tau}^{3}} \ln \frac{S}{K} \\
\frac{\partial d_{-}}{\partial \tau} &= \frac{\partial (d_{+} - \sigma \sqrt{\tau})}{\partial \tau} = \frac{\mu}{2\sigma \sqrt{\tau}} - \frac{\sigma}{4\sqrt{\tau}} - \frac{1}{2\sigma \sqrt{\tau}^{3}} \ln \frac{S}{K}
\end{align*}
\]  

(22)

The theta can then be derived using identity \( \dot{\mathcal{P}}S\Phi(d_{+}) = PK\Phi(d_{-}) \), that is

\[
\begin{align*}
\theta &= \frac{\partial \mathcal{B}}{\partial t} = \omega \dot{\mathcal{P}}S\Phi(\omega d_{+}) - \dot{\mathcal{P}}S\Phi(d_{+}) \frac{\partial d_{+}}{\partial \tau} - \omega r PK\Phi(\omega d_{-}) + PK\Phi(d_{-}) \frac{\partial d_{-}}{\partial \tau} \\
&= \omega \dot{\mathcal{P}}S\Phi(\omega d_{+}) - \omega r PK\Phi(\omega d_{-}) - \dot{\mathcal{P}}S\Phi(d_{+}) \frac{\sigma}{2\sqrt{\tau}}
\end{align*}
\]  

(23)

3.2.2. Gamma

Spot (forward) Gamma \( \Gamma \) is the first derivative of the spot (forward) delta \( \Delta \) with respect to the underlying spot \( S_{t} \) (forward \( F_{t,T} \)), or equivalently the second derivative of the present (forward) value of the option with respect to the spot (forward)

\[
\Gamma = \frac{\partial^{2} \mathcal{B}}{\partial S^{2}} = \frac{\partial \Delta}{\partial S} = \frac{\dot{\mathcal{P}}\Phi(d_{+})}{S \sigma \sqrt{\tau}}, \quad \Gamma_{F} = \frac{\partial^{2} \mathcal{B}}{\partial F^{2}} = \frac{\partial \Delta_{F}}{\partial F} = \frac{\Phi(d_{+})}{F \sigma \sqrt{\tau}}
\]  

(24)

The call and the put option with an equal strike have the same gamma sensitivity.

3.2.3. Vega
Vega \( \mathcal{V} \) is the first derivative of the option price with respect to the volatility \( \sigma \). Let’s first derive

\[
\frac{\partial d_+}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left( \frac{1}{\sigma \sqrt{t}} \ln \left( \frac{F}{K} \right) + \frac{\sigma \sqrt{t}}{2} \right) = -\frac{1}{\sigma^2 \sqrt{t}} \ln \left( \frac{F}{K} \right) + \frac{\sigma \sqrt{t}}{2} = - \frac{d_+}{\sigma} + \frac{\sqrt{t}}{\sigma} = - \frac{d_-}{\sigma}
\]

(25)

\[
\frac{\partial d_-}{\partial \sigma} = \frac{\partial}{\partial \sigma} (d_+ - \sigma \sqrt{t}) = \frac{\partial d_+}{\partial \sigma} - \sqrt{t} = - \frac{d_+}{\sigma}
\]

Therefore, we have

\[
\mathcal{V} = \frac{\partial \mathcal{B}}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_+) \partial d_+}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_-) \partial d_-}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_+) d_+ - d_-}{\sigma} = \frac{\mathcal{P} \mathcal{S} \sqrt{\tau} \phi(d_+)}{S \sigma \sqrt{t}}
\]

(26)

The call and the put option with an equal strike have the same vega sensitivity.

3.2.4. \textit{Vanna}

Vanna \( \mathcal{V}_S \) is the cross derivative of the option price with respect to the initial spot \( S_t \) and the volatility \( \sigma \). The Vanna can be derived as

\[
\mathcal{V}_S = \frac{\partial^2 \mathcal{B}}{\partial S \partial \sigma} = \frac{\partial \Delta_S}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_+) \partial d_+}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_-) \partial d_-}{\partial \sigma} = \frac{\partial \mathcal{P} \phi(d_+) d_+ - d_-}{\sigma} = \frac{\mathcal{V} d_+ d_-}{S \sigma \sqrt{t}}
\]

(27)

The call and the put option with an equal strike have the same vanna sensitivity.

3.2.5. \textit{Volga}

Volga \( \mathcal{V}_\sigma \) is the second derivative of the option price with respect to the volatility \( \sigma \)

\[
\mathcal{V}_\sigma = \frac{\partial^2 \mathcal{B}}{\partial \sigma^2} = \frac{\partial \mathcal{V}}{\partial \sigma} = \frac{\partial \mathcal{P} \sqrt{\tau} \phi(d_+) \partial d_+}{\partial \sigma} = \frac{\partial \mathcal{P} \sqrt{\tau} \phi(d_+) d_+ - d_-}{\sigma} = \frac{\mathcal{V} d_+ d_-}{\sigma}
\]

(28)

using the fact that

\[
\frac{\partial \phi(d_+)}{\partial d_+} = \frac{\partial}{\partial d_+} \left( \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{d_+^2}{2} \right) \right) = - \phi(d_+) d_+
\]

(29)

The call and the put option with an equal strike have the same volga sensitivity.

4. \textbf{FX Volatility Convention}

In liquid FX markets, Straddle, Risk Reversal and Butterfly are some of the most traded option strategies. It is convention that the markets usually quote volatilities instead of the direct prices of these
instruments, and typically express these volatilities as functions of delta, e.g. $\delta = 0.25$ or $0.1$, which are commonly referred to as the 25-Delta or the 10-Delta. Let’s define a general form of delta function $\Delta(\omega, K, \sigma)$, which can be any of the pips spot $\Delta$, pips forward $\Delta_F$, percentage spot $\Delta_p$ or percentage forward $\Delta_{pF}$. The $\delta$ in Black-Scholes model can be computed by the delta function $\Delta(\omega, K, \sigma)$ from a strike $K$ and a volatility $\sigma$. Providing a market consistent volatility smile $\sigma(K)$ at a maturity, there is a 1-to-1 mapping from $\delta$ to $K$ such that $\delta = \Delta(\omega, K, \sigma(K))$.

4.1. At-The-Money Volatility

FX markets quote the at-the-money volatility $\sigma_{atm}$ against a conventionally defined at-the-money strike $K_{atm}$. There are mainly two types of at-the-money definitions: ATM forward (ATMFWD) and ATM delta-neutral straddle (ATMDNS). A market consistent volatility smile $\sigma(K)$ must admit the fact that $\sigma(K_{atm}) = \sigma_{atm}$.

4.1.1. ATM Forward

In this definition, the at-the-money strike is set to the FX forward $F_{t,T}$

$$K_{atm} = F_{t,T}$$ (30)

This convention is used for currency pairs including a Latin American emerging market currency, e.g. MXN, BRL, etc. It may also apply to options with maturities longer than 10Y.

4.1.2. Delta Neutral Straddle

A delta-neutral straddle (DNS) is a straddle with zero combined call and put delta, such as

$$\Delta(1, K_{atm}, \sigma_{atm}) + \Delta(-1, K_{atm}, \sigma_{atm}) = 0$$ (31)

If the $\Delta(\omega, K, \sigma)$ is in the form of pips spot delta (12) or pips forward delta (16), the ATM strike $K_{atm}$ corresponding to the ATM volatility $\sigma_{atm}$ can be derived as

$$\Phi(d_+) - \Phi(-d_+) = 0 \Rightarrow \Phi(d_+) = 0.5 \Rightarrow K_{atm} = F \exp\left(\frac{1}{2} \sigma_{atm}^2 \tau\right)$$ (32)

Alternatively, if the $\Delta(K, \sigma, \omega)$ takes the form of percentage spot delta (14) or percentage forward delta (18), the ATM strike $K_{atm}$ can be derived as
\[ \Phi(d_-) - \Phi(-d_-) = 0 \implies \Phi(d_-) = 0.5 \implies K_{atm} = F \exp\left( -\frac{1}{2} \sigma_{atm}^2 \tau \right) \]  

(33)

The table below summarizes the ATMFWD and ATMDNS strikes with associated delta definitions

<table>
<thead>
<tr>
<th>Delta Convention</th>
<th>Delta Formula</th>
<th>ATMFWD Delta</th>
<th>ATMDNS Strike</th>
<th>ATMDNS Delta</th>
</tr>
</thead>
<tbody>
<tr>
<td>pips spot</td>
<td>( \omega \Phi(\omega d_+) )</td>
<td>( \omega \Phi\left(\omega \frac{\sigma_{atm} \sqrt{\tau}}{2}\right) )</td>
<td>( F \exp\left( \frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
<td>( \frac{1}{2} \omega \hat{p} )</td>
</tr>
<tr>
<td>pips forward</td>
<td>( \omega \Phi(\omega d_) )</td>
<td>( \omega \Phi\left(\omega \frac{\sigma_{atm} \sqrt{\tau}}{2}\right) )</td>
<td>( F \exp\left( -\frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
<td>( -\frac{1}{2} \omega )</td>
</tr>
<tr>
<td>percentage spot</td>
<td>( \omega \hat{p} \frac{K}{F} \Phi(\omega d_-) )</td>
<td>( \omega \hat{p} \Phi\left(-\omega \frac{\sigma_{atm} \sqrt{\tau}}{2}\right) )</td>
<td>( F \exp\left( -\frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
<td>( \frac{1}{2} \omega \hat{p} \exp\left( -\frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
</tr>
<tr>
<td>percentage forward</td>
<td>( \frac{K}{F} \Phi(\omega d_-) )</td>
<td>( \omega \Phi\left(-\omega \frac{\sigma_{atm} \sqrt{\tau}}{2}\right) )</td>
<td>( F \exp\left( -\frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
<td>( \frac{1}{2} \omega \exp\left( -\frac{\sigma_{atm}^2 \tau}{2} \right) )</td>
</tr>
</tbody>
</table>

It is evident that if the ATM strike is greater (smaller) than the forward, the market convention must be that deltas for that currency pair are quoted as pips (percentage) deltas [9].

4.2. Risk Reversal Volatility

FX markets quote the risk reversal volatility \( \sigma_{\delta RR} \) as a difference between the \( \delta \)-delta call and put volatilities. Providing a market consistent volatility smile \( \sigma(K) \), it is given by

\[ \sigma_{\delta RR} = \sigma(K_{\delta C}) - \sigma(K_{\delta P}) \]  

(34)

where \( \delta \)-delta smile strikes \( K_{\delta C} \) and \( K_{\delta P} \) can be inverted from the delta function such that

\[ \Delta(1, K_{\delta C}, \sigma(K_{\delta C})) = \delta, \quad \Delta(-1, K_{\delta P}, \sigma(K_{\delta P})) = -\delta \]  

(35)

4.3. Strangle Volatility

There are two types of strangle volatilities.

4.3.1. Market Strangle

Market strangle (MS, also known as brokers fly) is quoted as a single volatility \( \sigma_{\delta MS} \) for a delta \( \delta \). The \( \delta \)-delta market strangle strikes \( K_{MS, \delta C} \) and \( K_{MS, \delta P} \) for the call and put are both calculated in Black-Scholes model with a single constant volatility of \( \sigma_{atm} + \sigma_{\delta MS} \), such that at these strikes the call and put have deltas of \( \pm \delta \) respectively
\[ \Delta(1, K_{MS,\delta C}, \sigma_{atm} + \sigma_{\delta MS}) = \delta, \quad \Delta(-1, K_{MS,\delta P}, \sigma_{atm} + \sigma_{\delta MS}) = -\delta \]  

(36)

This gives the value of the market strangle in Black-Scholes model as

\[ V_{\delta MS} = \mathbb{B}(1, K_{MS,\delta C}, \sigma_{atm} + \sigma_{\delta MS}, \tau) + \mathbb{B}(-1, K_{MS,\delta P}, \sigma_{atm} + \sigma_{\delta MS}, \tau) \]  

(37)

This value must be satisfied by a market consistent volatility smile \( \sigma(K) \), such that the \( V'_{\delta MS} \) below must be equal to the \( V_{\delta MS} \)

\[ V'_{\delta MS} = \mathbb{B}(1, K_{MS,\delta C}, \sigma(K_{MS,\delta C}), \tau) + \mathbb{B}(-1, K_{MS,\delta P}, \sigma(K_{MS,\delta P}), \tau) \]  

(38)

Note that, at these strikes we generally have

\[ \Delta(1, K_{MS,\delta C}, \sigma(K_{MS,\delta C})) \neq \delta, \quad \Delta(-1, K_{MS,\delta P}, \sigma(K_{MS,\delta P})) \neq -\delta \]  

(39)

Providing a calibrated volatility smile \( \sigma(K) \) consistent with the market, it is easy to derive the market strangle volatility from the smile. The procedure takes the following steps

1. Choose an initial guess for \( \sigma_{\delta MS} \) (e.g. \( \sigma_{\delta MS} = \sigma_{\delta SS} \))
2. Compute the market strangle strikes \( K_{MS,\delta C} \) and \( K_{MS,\delta P} \) by (36)
3. Compute the strangle value \( V_{\delta MS} \) in (37) and the \( V'_{\delta MS} \) in (38)
4. If \( V'_{\delta MS} \) is close to \( V_{\delta MS} \) then the \( V_{\delta MS} \) is found, otherwise go to step 1 to repeat the iteration

4.3.2. Smile Strangle

Providing a market consistent volatility smile \( \sigma(K) \) is available, it is more intuitive to express the strangle volatility \( \sigma_{\delta SS} \) as

\[ \sigma_{\delta SS} = \frac{\sigma(K_{\delta C}) + \sigma(K_{\delta P})}{2} - \sigma(K_{atm}) \]  

(40)

This is called smile strangle volatility, where the smile strikes \( K_{\delta C} \) and \( K_{\delta P} \) are given by (35).

Given the market quoted \( \sigma_{atm}, \sigma_{\delta RR} \) and \( \sigma_{\delta MS} \), one can build a volatility smile \( \sigma(K) \) that is consistent with the market. The procedure takes the following steps

1. Preparation:
   - Determine the delta convention (e.g. pips or percentage, spot or forward)
• Determine the at-the-money convention (e.g. ATMFWD or ATMDNS) and its associated ATM strike $K_{atm}$

• Choose a parametric form for the volatility smile $\sigma(K)$ (e.g. Polynomial-in-Delta interpolation)

• Determine the market strangle strikes $K_{MS,\delta C}$ and $K_{MS,\delta P}$ by (36) using $\sigma_{atm} + \sigma_{\delta MS}$

• Compute the value of market strangle $V_{\delta MS}$ in (37)

2. Choose an initial guess for $\sigma_{\delta SS}$ (e.g. $\sigma_{\delta SS} = \sigma_{\delta MS}$)

3. Use $\sigma_{atm}$, $\sigma_{\delta RR}$ and $\sigma_{\delta SS}$ to find the best fit of $\sigma(K)$ such that with the smile strikes $K_{\delta C}$ and $K_{\delta P}$ given by (35), we have

$$\sigma(K_{atm}) = \sigma_{atm}$$

$$\sigma(K_{\delta C}) - \sigma(K_{\delta P}) = \sigma_{\delta RR}$$

$$\frac{\sigma(K_{\delta C}) + \sigma(K_{\delta P})}{2} - \sigma(K_{atm}) = \sigma_{\delta SS}$$

(41)

4. Compute the value of the market strangle $V'_{\delta MS}$ in (38) with the market strangle strikes $K_{MS,\delta C}$ and $K_{MS,\delta P}$ using the $\sigma(K)$ fitted in step 3.

5. If $V'_{\delta MS}$ is close to the true market strangle $V_{\delta MS}$ then the $\sigma(K)$ is found, otherwise go to step 2 to repeat the iteration.

5. **Volatility Surface Construction**

Table 2 presents an example of ATM, risk reversal and smile strangle volatilities at a series of maturities. Each maturity may associate with different ATM and delta conventions. In previous section, we have shown how to extract the five volatilities, at ±10D ±25D and ATM respectively, from market quotes for each maturity subject to its associated market convention. It is often desired to have a volatility surface, so that an implied volatility at arbitrary delta/strike and maturity can be interpolated from the surface.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>ATM Convention</th>
<th>Delta Convention</th>
<th>ST10D</th>
<th>ST25D</th>
<th>ATM</th>
<th>RR25D</th>
<th>RR10D</th>
</tr>
</thead>
<tbody>
<tr>
<td>1M</td>
<td>ATMDNS</td>
<td>Spot Percentage</td>
<td>0.73%</td>
<td>0.28%</td>
<td>9.13%</td>
<td>-1.13%</td>
<td>-2.09%</td>
</tr>
</tbody>
</table>
5.1. Smile Interpolation

There are many ways to perform smile interpolation. Polynomial-in-Delta is one of the simple and widely used methods. It employs a 4\textsuperscript{th} order polynomial which allows a perfect fit to five volatilities of a smile (or a 2\textsuperscript{nd} order polynomial if just fitting to three volatilities). The parameterization is as follows

\[
\ln \sigma(K) = \sum_{j=0}^{4} a_j x(K)^j, \quad x(K) = M(K) - M(Z) \tag{42}
\]

where \(a_j\)'s are the coefficients to be calibrated (exactly) to the market volatilities. The function \(M(\cdot)\) provides a measure of moneyness that often takes the form

\[
M(K) = \Phi \left( \frac{1}{\nu \sqrt{T}} \ln \frac{K}{\Lambda} \right) \tag{43}
\]

where \(\Lambda\) can be the forward \(F\) or the at-the-money strike \(K_{atm}\). The parameter \(Z\) in (42) can be chosen to be the \(F\) or the \(K_{atm}\) such that the \(x(K)\) provides a measure of distance in moneyness from the \(Z\). The parameter \(\nu\) in (43) may simply take \(\nu = \sigma_{atm}\). However, to be more adaptive, one may choose \(\nu = \sigma(K)\), together with which the (42) must then be solved iteratively for the \(\sigma(K)\). This interpolation is named after the fact that the measure of moneyness (43) is similar to the definition of forward delta (16).

The calibration of the coefficients \(a_j\) is straightforward. From previous discussion, we are able to retrieve 5 volatility-strike pairs \((\sigma_i, K_i)\) for \(i = 1, \ldots, 5\) at a given maturity from market quotes, i.e. volatilities at strikes corresponding to \(\pm 10D\), \(\pm 25D\) and ATM subject to proper delta and ATM
conventions. Based on the 5 volatilities, we are able to form a full rank linear system from (42), which can then be solved for the coefficients $a_j$’s.

5.2. Temporal Interpolation

The most commonly used temporal interpolation assumes a flat forward volatility in time. This is equivalent to a linear interpolation in total variance. For example, if we have $\sigma_{\text{atm}}(p)$ and $\sigma_{\text{atm}}(q)$ at maturities $p$ and $q$ respectively, subject to the same ATM and delta convention, we may interpolate an ATM volatility at a time $t$ for $p < t < q$ by the formula

$$
\sigma_{\text{atm}}^2(t)t = \frac{q - t}{q - p} \sigma_{\text{atm}}^2(p)p + \frac{t - p}{q - p} \sigma_{\text{atm}}^2(q)q
$$

(44)

The temporal interpolation in $\pm10D$ and $\pm25D$ volatilities may follow the same manner.

5.3. Volatility Surface by Standard Conventions

Table 2 shows that the market convention on ATM and delta style may vary from one maturity to another. Such jumps in convention introduce inconsistency in definition of the ATM strikes and $\delta$-deltas at different maturities. We must choose a consistent set of smile conventions for marking an ATM strike and $\delta$-deltas strikes at all maturities [10]. A pragmatic choice is to use delta-neutral ATM and forward pips delta as the standard conventions. We may convert the volatility-strike pairs $(\sigma_i, K_i)$ for $i = 1, \cdots, 5$ at maturity $t$ to $(\bar{\sigma}_i, \bar{K}_i)$ such that the $(\bar{\sigma}_i, \bar{K}_i)$ at all maturities follow the same unified standard conventions. The temporal interpolation is then performed on the standardized volatilities, e.g. between $\bar{\sigma}_{25D}(p)$ and $\bar{\sigma}_{25D}(q)$ to get a $25D$ volatility at an interim time $t$ for $p < t < q$. Following the same manner, five volatilities $\bar{\sigma}_i(t)$ can be obtained, along with their associated strikes $\bar{K}_i(t)$ (inverted from the $\delta$-delta values given the standard conventions we have chosen). The last step is then to build a smile based on the $\bar{\sigma}_i(t)$ and $\bar{K}_i(t)$ for strike interpolation.

The conversion from $(\sigma_i, K_i)$ to $(\bar{\sigma}_i, \bar{K}_i)$ can be trivial. We must at first fit a smile $\sigma(K)$ to the $(\sigma_i, K_i)$. To be consistent across maturities, it is ideal to choose $Z = \bar{K}_{\text{atm}}$ in (42). This requires finding iteratively the $\bar{K}_{\text{atm}}$ and the $\bar{\sigma}_{\text{atm}} = \sigma(\bar{K}_{\text{atm}})$ that conform to the standard ATM and delta conventions.
(e.g. equation (32) must be satisfied). Once the smile \( \sigma(K) \) is available, it is trivial to find all the \((\bar{\sigma}_i, \bar{\gamma}_i)\) at the \( \pm 10D \) and \( \pm 25D \) subject to the standard conventions.

6. **The Vanna-Volga Method**

The vanna-volga method is a technique for pricing first-generation FX exotic products (e.g. barriers, digitals and touches, etc.). The main idea of vanna-volga method is to adjust the Black-Scholes theoretical value (TV) of an option by adding the smile cost of a portfolio that hedges three main risks associated to the volatility of the option: the vega, vanna and volga.

6.1. **Vanna-Volga Pricing**

Suppose there exists a portfolio \( H \) with a long position in an exotic trade \( X \), a short position in \( \Delta \) amount of the underlying spot \( S \), and short positions in \( \omega_1 \) amount of instrument \( A_1 \), \( \omega_2 \) amount of instrument \( A_2 \) and \( \omega_3 \) amount of instrument \( A_3 \). The hedging instruments \( A_i \)'s can be the straddle, risk reversal and butterfly, as they are liquidly traded in FX markets and they carry mainly vega, vanna and volga risks respectively that can be used to hedge the volatility risks of the trade \( X \). By construction, the price of the portfolio and its dynamics must follow

\[
H = X - \Delta S - \sum_{i=1}^{3} \omega_i A_i, \quad dH = dX - \Delta dS - \sum_{i=1}^{3} \omega_i dA_i
\]  

We may estimate the Greeks in Black-Scholes model and further express the price dynamics in terms of the stochastic spot \( S \) and flat volatility \( \sigma \). By Ito’s lemma, we have

\[
dH = \left( \frac{\partial X}{\partial t} - \sum_{i=1}^{3} \omega_i \frac{\partial A_i}{\partial t} \right) dt + \left( \frac{\partial X}{\partial S} - \Delta - \sum_{i=1}^{3} \omega_i \frac{\partial A_i}{\partial S} \right) dS + \frac{1}{2} \left( \frac{\partial^2 X}{\partial S^2} - \sum_{i=1}^{3} \omega_i \frac{\partial^2 A_i}{\partial S^2} \right) dS dS
\]

\[
+ \left( \frac{\partial X}{\partial \sigma} - \sum_{i=1}^{3} \omega_i \frac{\partial A_i}{\partial \sigma} \right) d\sigma + \frac{1}{2} \left( \frac{\partial^2 X}{\partial \sigma^2} - \sum_{i=1}^{3} \omega_i \frac{\partial^2 A_i}{\partial \sigma^2} \right) d\sigma d\sigma
\]

Choosing the \( \Delta \) and the weights \( \omega_i \) so as to zero out the coefficients of \( dS \), \( d\sigma \), \( d\sigma d\sigma \) and \( dS d\sigma \), the portfolio is then *locally* risk free at time \( t \) (given that the gamma and other higher order risk factors can be
ignored) and must have a return at risk free rate. Therefore, when the flat volatility is stochastic and the options are valued in Black-Scholes model, we can still have a locally perfect hedge. The perfect hedge in the three volatility risks implies that the following linear system must be satisfied

\[
\begin{pmatrix}
\text{vega}(X) \\
\text{vanna}(X) \\
\text{volga}(X)
\end{pmatrix} =
\begin{pmatrix}
\text{vega}(A_1) & \text{vega}(A_2) & \text{vega}(A_3) \\
\text{vanna}(A_1) & \text{vanna}(A_2) & \text{vanna}(A_3) \\
\text{volga}(A_1) & \text{volga}(A_2) & \text{volga}(A_3)
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\]

(47)

This perfect hedging is under an assumption of flat volatility. Due to non-flat nature of the volatility surface, additional cost between \(A_i(\sigma_{\text{smile}})\) and \(A_i(\sigma_{\text{flat}})\) must be accounted into the price of the trade \(X\) to fulfil the hedging. As a result, the vanna-volga price \(X_{VV}\) of the trade \(X\) is computed as follows

\[
X_{VV}(\sigma_{\text{smile}}) = X_{TV}(\sigma_{\text{flat}}) + \sum_{i=1}^{3} \omega_i (A_i(\sigma_{\text{smile}}) - A_i(\sigma_{\text{flat}}))
\]

(48)

where \(X_{TV}(\sigma_{\text{flat}})\) is the theoretical Black-Scholes value using a flat volatility (e.g. \(\sigma_{\text{flat}} = \sigma_{\text{atm}}\)), \(A_i(\sigma_{\text{smile}})\) and \(A_i(\sigma_{\text{flat}})\) are the prices of the hedging instrument valued with a volatility smile and a flat volatility respectively.

6.2. Smile Interpolation

The vanna-volga method may also serve a purpose of interpolating a volatility smile based on the market quoted at-the-money volatility \(\sigma_{\text{atm}}\), the \(\delta\)-delta risk reversal volatility \(\sigma_{\delta\text{RR}}\), and lastly the \(\delta\)-delta smile strangle volatility \(\sigma_{\delta\text{SS}}\) (converted from market strangle volatility \(\sigma_{\delta\text{MS}}\) by the method in section 4.3.2). From the relationship in (41), we can derive the following quantities

<table>
<thead>
<tr>
<th>Strikes</th>
<th>Implied Volatilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_1 = K_{\delta P})</td>
<td>(\sigma_1 = \sigma(K_{\delta P}) = \sigma_{\text{atm}} + \sigma_{\delta\text{SS}} - \frac{\sigma_{\delta\text{RR}}}{2})</td>
</tr>
<tr>
<td>(K_2 = K_{\text{atm}})</td>
<td>(\sigma_2 = \sigma(K_{\text{atm}}) = \sigma_{\text{atm}})</td>
</tr>
<tr>
<td>(K_3 = K_{\delta C})</td>
<td>(\sigma_3 = \sigma(K_{\delta C}) = \sigma_{\text{atm}} + \sigma_{\delta\text{SS}} + \frac{\sigma_{\delta\text{SS}}}{2})</td>
</tr>
</tbody>
</table>

where the ATM strike \(K_{\text{atm}}\) is given by the at-the-money convention, and the \(\delta\)-delta smile strikes \(K_{\delta C}\) and \(K_{\delta P}\) are solved by (35).
We will follow a similar analysis as in section 6.1. Suppose we have a perfect hedged portfolio $P$ that consists of a long position in a call option $X$ with an arbitrary strike $K$, a short position in $\Delta$ amount of spot $S$, and three short positions in $\omega_i$ amount of call options $A_i$ with strikes $K_1 = K_{\delta P}$, $K_2 = K_{atm}$ and $K_3 = K_{\delta C}$. The perfect hedge in the three volatility risks admits that the following linear system must be satisfied

\[
\begin{pmatrix}
\text{vega}(X) \\
\text{vanna}(X) \\
\text{volga}(X)
\end{pmatrix} =
\begin{pmatrix}
\text{vega}(A_1) & \text{vega}(A_2) & \text{vega}(A_3) \\
\text{vanna}(A_1) & \text{vanna}(A_2) & \text{vanna}(A_3) \\
\text{volga}(A_1) & \text{volga}(A_2) & \text{volga}(A_3)
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\]

(49)

where these volatility sensitivities can be estimated in Black-Scholes model assuming a flat volatility flat volatility $\sigma$ (usually we choose $\sigma = \sigma_{atm}$). Plugging the closed form Black-Scholes vega, vanna and volga in (26) (27) and (28) respectively, the (49) becomes

\[
\mathcal{V}(K)
\begin{pmatrix}
\frac{1}{d_+(K)} \\
\frac{1}{d_-(K)}
\end{pmatrix} =
\begin{pmatrix}
\mathcal{V}(K_1) & \mathcal{V}(K_2) & \mathcal{V}(K_3) \\
\mathcal{Vd}_+(K_1) & \mathcal{Vd}_+(K_2) & \mathcal{Vd}_+(K_3) \\
\mathcal{Vd}_-(K_1) & \mathcal{Vd}_-(K_2) & \mathcal{Vd}_-(K_3)
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_3
\end{pmatrix}
\]

(50)

where $\mathcal{Vd}_+(K)$ is short for $\mathcal{V}(K)d_+(K)d_-(K)$. By inverting the linear system, there is a unique solution of $\omega$ for the strike $K$, such that

\[
\omega_1 = \frac{\mathcal{V}(K) \ln K_2}{\mathcal{V}(K_1) \ln K_2} \frac{K_3}{K_1},
\quad
\omega_2 = \frac{\mathcal{V}(K) \ln K_3}{\mathcal{V}(K_2) \ln K_3} \frac{K_1}{K_2},
\quad
\omega_3 = \frac{\mathcal{V}(K) \ln K_1}{\mathcal{V}(K_3) \ln K_1} \frac{K_2}{K_3}
\]

(51)

A “smile-consistent” volatility $v$ (i.e. a Black Scholes volatility implied from the price by the vanna-volga method) for the call with the strike $K$ is then obtained by adding to the Black-Scholes price the cost of implementing the above hedging at prevailing market prices, that is

\[
\mathcal{C}(K, v) = \mathcal{C}(K, \sigma) + \sum_{i=1}^{3} \omega_i (\mathcal{C}(K_i, \sigma) - \mathcal{C}(K_i, \sigma))
\]

(52)

where the function $\mathcal{C}(K, \sigma)$ stands for the Black-Scholes call option price with strike $K$ and flat volatility $\sigma$. 
A market implied volatility curve can then be constructed by inverting (52), for each considered $K$. Here we introduce an approximation approach. By taking the first order expansion of (52) in $\sigma$, that is

$$C(K, \nu) \approx C(K, \sigma) + \sum_{i=1}^{3} \omega_i (\sigma_i - \sigma) V(K_i)$$ (53)

Substituting $\omega_i$ with the results in (51) and using the fact that $V(K) = \sum_{i=1}^{3} \omega_i V(K_i)$, we have

$$C(K, \nu) \approx C(K, \sigma) + V(K) \left( \sum_{i=1}^{3} y_i \sigma_i - \sigma \right) \approx C(K, \sigma) + V(K) (\bar{\nu} - \sigma) \Rightarrow \bar{\nu} \approx \sum_{i=1}^{3} y_i \sigma_i \quad (54)$$

where $\bar{\nu}$ is the first order approximation of the implied volatility $\nu$ for strike $K$, and the coefficients $y_i$ are given by

$$y_1 = \frac{\ln K_2 - \ln K_1}{\ln K_3 - \ln K_1}, \quad y_2 = \frac{\ln K - \ln K_2}{\ln K_3 - \ln K_2}, \quad y_3 = \frac{\ln K - \ln K_1}{\ln K_2 - \ln K_1} \quad (55)$$

This shows that the implied volatility $\nu$ can be approximated by a linear combination of the three smile volatilities $\sigma_i$.

A more accurate second order approximation, which is asymptotically constant at extreme strikes, can be obtained by expanding the (52) at second order in $\sigma$

$$C(K, \nu) \approx C(K, \sigma) + V(K) (\bar{\nu} - \sigma) + \frac{1}{2} V_{\sigma}(K) (\bar{\nu} - \sigma)^2$$

$$\approx C(K, \sigma) + \sum_{i=1}^{3} \omega_i \left( V(K_i) (\sigma_i - \sigma) + \frac{1}{2} V_{\sigma}(K_i) (\sigma_i - \sigma)^2 \right)$$

$$\Rightarrow V(K) (\bar{\nu} - \sigma) + \frac{V_{d+}d_{-}(K)}{2\sigma} (\bar{\nu} - \sigma)^2$$

$$\approx V(K) \sum_{i=1}^{3} y_i \sigma_i - V(K) \sigma + \frac{V(K)}{2\sigma} \sum_{i=1}^{3} y_i d_+(K_i) (\sigma_i - \sigma)^2$$
\[ \Rightarrow \frac{d_+ d_-(K)}{2\sigma} (\bar{v} - \sigma)^2 + (\bar{v} - \sigma) - \left( \bar{v} - \sigma + \frac{\sum_{l=1}^{3} y_l d_+ d_-(K_l)(\sigma_l - \sigma)^2}{2\sigma} \right) \approx 0 \]

Solving the quadratic equation in (56) gives the second order approximation

\[ \bar{v} \approx \sigma + \frac{-\sigma + \sqrt{\sigma^2 + (2\sigma(\bar{v} - \sigma) + \sum_{l=1}^{3} y_l d_+ d_-(K_l)(\sigma_l - \sigma)^2)d_+ d_-(K)}}{d_+ d_-(K)} \quad (57) \]

where \( d_+ d_-(K) \) stands for \( d_+ (K) d_- (K) \) that is evaluated with a flat volatility \( \sigma \).
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