Calibration of SABR Stochastic Volatility Model

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1. INTRODUCTION

SABR model is a CEV model augmented by stochastic volatility that assumes the forward rate evolves under the associated forward (terminal) measure $Q^T$

$$dF_{t,T} = \alpha_t F_t^{\beta} dW_t, \quad d\alpha_t = \nu \alpha_t dZ_t, \quad dW_t dZ_t = \rho dt$$

for time $t$ between initial time $s$ and maturity $T$. The $F_{t,T}$ is a forward rate process with initial value $F_{s,T} = f$. The $\alpha_t$ is the stochastic volatility with initial value $\alpha_s = a$. The parameter $a$ cannot be observed from the market, however it can be derived analytically from the at-the-money implied volatility as we shall see in due course. The factor $\nu$ is known as the volatility of volatility, which adjusts the degree of volatility clustering in time. The parameter $\beta \in [0,1]$ controls the relationship between the forward rate and the at-the-money volatility. A $\beta < 1$ (“non-lognormal” case) leads to skews in the implied volatilities. In the case of $\beta \approx 1$, if the market were to move up or down, the level of the at-the-money volatility would not be significantly affected, whereas when $\beta < 1$ the volatility increases as forward rate falls (i.e. volatility and forward move in opposite direction). The closer to 0 the more pronounced would be this effect. The correlation parameter $\rho$ plays a similar role as the $\beta$ does. It defines how the market moves in sync with the volatility dynamics. The model parameters $\nu, a, \beta$ and $\rho$ are all assumed to be deterministic and time invariant.

2. ASYMPTOTIC SOLUTION BY HAGAN ET AL.

Using singular perturbation techniques, Hagan et al. [1] provide a closed form asymptotic solution (up to the accuracy of a series expansion) for prices of vanilla instruments. The value of a vanilla option under the SABR model is given by the appropriate Black formula provided that the correct Black implied volatility is used. Given the forward initial value $F_{s,T} = f$ and the expiry time $\tau = T - s$, the Black implied volatility $\sigma$ can be derived as a function of strike price $K$ from a given set of SABR parameters $a, \nu, \beta$ and $\rho$ by
\[ \sigma_{a,v,\beta,\rho}(K) = \frac{\zeta}{\chi} \cdot a \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24 p^2} + \frac{\rho \beta v a}{4p} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right) \]

\[ p = (f/K)^{1-\beta}, \quad q = \ln \frac{f}{K}, \quad \zeta = \frac{vpq}{a}, \quad \chi = \ln \left( \frac{\sqrt{1 - 2 \rho \zeta + \zeta^2 + \zeta - \rho}}{1 - \rho} \right) \]

where

\[ \sigma_f = \sigma_{a,v,\beta,\rho}(f) = \frac{a}{f^{1-\beta}} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24 p^2} + \frac{\rho \beta v a}{4p} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right) \]

When \( K = f \), equation (2) reduces to give the at-the-money implied volatility

\[ \sigma_f = \sigma_{a,v,\beta,\rho}(f) = \frac{a}{f^{1-\beta}} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24 p^2} + \frac{\rho \beta v a}{4p} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right) \]

The (3) shows that there exists a relationship

\[ \ln \sigma_f = \ln a - (1 - \beta) \ln f + \cdots \]

It indicates that the value of \( \beta \) can be estimated from a log-log regression of \( \sigma_f \) and \( f \) with historical data by ignoring terms involving \( \tau \). Alternatively, since the parameters \( \beta \) and \( \rho \) in SABR model control the distribution function in similar ways (i.e. both control the skewness of the distribution), the redundancy between the two parameters allows one to calibrate the model by fixing \( \beta \) to an assumption (e.g. \( \beta = 0.5 \)).

The decision is often made on the basis of market experience. The remaining parameters \( a, v \) and \( \rho \) have different effects on the volatility curve. The parameter \( a \) mainly controls the overall magnitude of the curve, the \( v \) controls how much smile (i.e. convexity) the curve exhibits and the \( \rho \) controls the curve’s skew.

As shown in (3) the parameter \( a \) has a functional form with the at-the-money volatility \( \sigma_f \).

Inverting the equation gives the value of \( a \) as a root of a cubic equation if the \( v \) and \( \rho \) are known (in general, the smallest positive root would be taken if there were three real roots)

\[ \frac{(1 - \beta)^2 \tau}{24 f^2(1-\beta)} a^3 + \frac{\rho \beta v \tau}{4 f^{1-\beta} a^2} + \left( 1 + \frac{(2 - 3 \rho^2) v^2 \tau}{24} \right) a - \sigma_f f^{1-\beta} = 0 \]

This indicates that in SABR model we only need to calibrate \( \rho \) and \( v \) to implied volatility surface, providing that the value of \( \beta \) is prescribed and the at-the-money implied volatility \( \sigma_f \) is given. The
calibration is performed at each maturity of the volatility surface by minimizing the objective function defined as a sum of squared residuals (or sum of vega weighted squared residuals)

$$\argmin_{\nu, \rho} \sum_{i=1}^{N} \left( \sigma_{K_i}^{mkt} - \sigma_{K_i}^{SABR(f, v, a, \beta, \rho)} \right)^2$$

(6)

Various nonlinear optimization routines can be used to carry out the calibration, for example, Levenberg-Marquardt method or Nelder-Mead simplex method.

3. **Correction to Hagan et al. Solution**

The $\zeta$ in (2) was defined in equation (A.57c) in [1]. When assuming CEV model for the forward process (i.e. $C(F) = F^\beta$), it has the form

$$\zeta = \frac{\nu}{a} \int_F^f \frac{dF}{C(F)} = \frac{\nu}{a} \int_K^f \frac{du}{u^\beta} = \frac{\nu}{a} \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} = \frac{\nu}{a} \eta, \quad \eta = \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta}$$

(7)

By expanding, we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^y - e^{-y} = 2y \left( 1 + y^2 \frac{1}{3!} + y^4 \frac{1}{5!} + \cdots \right)$$

$$\Rightarrow p \left( e^{\frac{1-\beta}{2}q} - e^{-\frac{1-\beta}{2}q} \right) = p(1-\beta)q \left( 1 + \frac{(1-\beta)^2q^2}{24} + \frac{(1-\beta)^4q^4}{1920} + \cdots \right),$$

for $y = \frac{1-\beta}{2} q, \quad p = (fK)^{1-\beta}, \quad q = \ln \frac{f}{K}$

$$\Rightarrow f^{1-\beta} - K^{1-\beta} = (1-\beta)pq \left( 1 + \frac{(1-\beta)^2q^2}{24} + \frac{(1-\beta)^4q^4}{1920} + \cdots \right)$$

$$\Rightarrow \eta = pq \left( 1 + \frac{(1-\beta)^2q^2}{24} + \frac{(1-\beta)^4q^4}{1920} + \cdots \right)$$

Thus $\zeta$ can be written as

$$\zeta = \frac{\nu}{a}pq \left( 1 + \frac{(1-\beta)^2q^2}{24} + \frac{(1-\beta)^4q^4}{1920} + \cdots \right)$$

(9)

Clearly the expression of $\zeta$ used in (2) is just an approximation of (9) truncating all higher order terms of $q$. This leads to a correction to the original Hagan et al. solution proposed by Obloj [2] in 2008, where he
uses $\zeta = \frac{v}{a} \eta$ in (7) to calculate $\zeta$ rather than $\zeta = \frac{v}{a} pq$ in (2). Plugging the $\zeta$ in (9) into the original Hagan et al. solution (2), we get the improved implied volatility formula

$$\sigma_{\alpha, \nu, \beta, \rho}(K) = \frac{pq}{\chi} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24p^2} + \frac{\rho \beta v a}{4p} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right)$$

where

$$p = (fK)^{\frac{1-\beta}{2}}, \quad q = \ln \frac{f}{K}, \quad \zeta = \frac{\nu f^{1-\beta} - K^{1-\beta}}{1 - \beta}, \quad \chi = \ln \left( \frac{\sqrt{1 - 2 \rho \zeta + \zeta^2 + \zeta - \rho}}{1 - \rho} \right)$$

Two special cases must be addressed. First, when $K \to f$, we get the at-the-money volatility

$$\lim_{K \to f} \sigma_{\alpha, \nu, \beta, \rho}(K) = \frac{a}{f^{1-\beta}} \left( 1 + \left( \frac{(1 - \beta)^2 a^2}{24f^{2(1-\beta)}} + \frac{\rho \beta v a}{4f^{1-\beta}} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right)$$

knowing that (or simply derived from the series expansion in (8))

$$\lim_{K \to f} \frac{f^{1-\beta} - K^{1-\beta}}{(1 - \beta)q} = \lim_{K \to f} \frac{-(1 - \beta)K^{-\beta}}{-(1 - \beta) \frac{1}{K}} = f^{1-\beta}, \quad \lim_{K \to f} \frac{\zeta}{\chi} = 1$$

$$\Rightarrow \lim_{K \to f} \frac{\nu q}{\chi} = \lim_{K \to f} \frac{\zeta}{\chi} \frac{(1 - \beta)q}{f^{1-\beta} - K^{1-\beta}} = \frac{a}{f^{1-\beta}}$$

Second, when $\beta \to 1$, we have the implied volatility in lognormal SABR

$$\lim_{\beta \to 1} \sigma_{\alpha, \nu, \beta, \rho}(K) = \frac{\nu q}{\chi} \left( 1 + \left( \frac{\rho v a}{4} + \frac{(2 - 3 \rho^2) v^2}{24} \right) \tau \right)$$

using the fact that

$$\lim_{\beta \to 1} \frac{f^{1-\beta} - K^{1-\beta}}{1 - \beta} = \lim_{\beta \to 1} \frac{-f^{1-\beta} \ln f + K^{1-\beta} \ln K}{-1} = \ln \frac{f}{K} = q \Rightarrow \lim_{\beta \to 1} \zeta = \frac{\nu q}{a}$$

4. **Advanced Analytics for the SABR Model**

Consider the SABR process

$$dF_t = F_t^{\rho} v_t dW_{1,t}, \quad dv_t = \gamma v_t dW_{2,t}, \quad dW_{1,t} dW_{2,t} = \rho dt$$

It is useful to transform the SABR rate process into a stochastic volatility Bessel process define as

$$Q_t = \frac{F_t^{\rho}}{1 - \beta}$$
The process $Q_t$ satisfies
\[
dQ_t = \left( \nu + \frac{1}{2} \frac{\nu^2}{Q_t} \right) dt + \nu_t dW_{1,t}, \quad d\nu_t = \gamma \nu_t dW_{2,t}, \quad dW_{1,t} dW_{2,t} = \rho dt
\] (17)

With the Bessel index
\[
\nu = -\frac{1}{2(1 - \beta)}
\] (18)

4.1. Zero Correlation Formulas

By change of variables
\[
V = \frac{\nu}{\gamma}, \quad V_0 = \frac{\nu_0}{\gamma}, \quad t \rightarrow \gamma^2 t
\] (19)

We can set $\gamma = 1$ to obtain the normalized form of the SABR evolution
\[
dF_t = F_t^{\beta} V_t dW_{1,t}, \quad dV_t = V_t dW_{2,t}, \quad dW_{1,t} dW_{2,t} = \rho dt
\] (20)

And the transformed Bessel process $Q_t$ satisfies
\[
dQ_t = \left( \nu + \frac{1}{2} \frac{\nu^2}{Q_t} \right) dt + \nu_t dW_{1,t}, \quad d\nu_t = \nu_t dW_{2,t}, \quad dW_{1,t} dW_{2,t} = \rho dt
\] (21)

The assumption of zero correlation permits “absorbing” the stochastic volatility $V_t$ into the new stochastic time $\tau$ for initial time $t = 0$
\[
\tau(t) = \int_0^t V_u^2 du
\] (22)

Indeed, the new process
\[
 dB_{1,\tau} = V_t dW_{1,t}
\] (23)

is a Brownian motion in time $\tau$ for $dB_{1,\tau} dB_{1,\tau} = V_t^2 dt = d\tau$. The process $B_{1,\tau}$ also remains uncorrelated with $W_{2,t}$. Denote the process $Q_t$ measured in the new time $\tau$ by $R_\tau \equiv Q_t$. Its governing SDE looks like
\[
dR_\tau = \left( \nu + \frac{1}{2} \frac{1}{R_\tau} \right) d\tau + dB_{1,\tau}
\] (24)

In other words, the process $R_\tau$ is a Bessel process with index $\nu$ given in (18).

4.1.1. The marginal PDF
The marginal distribution of the Bessel process $Q_t$ is defined as

$$p(t, q|q_0, V_0) = \mathbb{P}[Q_t \in dq|q_0, V_0] = \mathbb{E}[\delta(Q_t - q)|q_0, V_0] = \mathbb{E}[\mathbb{E}[\delta(R_t - q)|q_0]|V_0]$$  \hspace{1cm} (25)$$

where $\delta(\cdot)$ is the Dirac delta function. Since $B_{t,\tau}$ remains uncorrelated with $W_{2,t}$, the inner mean is the single point PDF of Bessel process with negative index $\nu$ and absorbing boundary condition

$$p(\tau, q|q_0) = \frac{1}{(1 - \beta)\tau} q_0$$  \hspace{1cm} (26)$$

$$\nu = -\frac{1}{2(1 - \beta)}$$  

5. **Correction to Hagan et al. Solution**

$$dF_{t,T} = \alpha_tF_{t,T}^\rho dW_t, \quad d\alpha_t = \nu \alpha_t dZ_t, \quad dW_t dZ_t = \rho dt$$  \hspace{1cm} (27)$$

Let’s write in (1)

$$dW = \rho dZ + \bar{\rho} dB, \quad dZdB = 0$$  \hspace{1cm} (28)$$

We can write

$$\int_s^t \alpha_u dW_u = \rho \int_s^t \alpha_u dZ_u + \bar{\rho} \int_s^t \alpha_u dB_u$$  \hspace{1cm} (29)$$

Using the 2nd equation in (1), the first integral is

$$\rho \int_s^t \alpha_u dZ_u = \frac{\rho}{\nu} (\alpha_t - \alpha_s)$$  \hspace{1cm} (30)$$

And

$$\alpha_t = \alpha_s \exp\left(\nu(Z_t - Z_s) - \frac{1}{2} \nu^2(t - s)\right)$$  \hspace{1cm} (31)$$

To approximate the 2nd integral in (29), we use the fact that conditional on the $\alpha_t$ path, this integral will have mean 0 and variance

$$\Sigma_{s,t} = \int_s^t \alpha_u du \approx \bar{\alpha}_t^2 t$$  \hspace{1cm} (32)$$
due to the independence between $Z$ and $B$. We approximate $\Sigma_{s,t}$ by a quantity dependent on the final path value $\alpha_t$, or equivalently $Z_t$ only, namely by \( \bar{\alpha}_t^2 t \) where

\[
\bar{\alpha}_t = \alpha_s \exp \left( \bar{\nu}(Z_t - Z_s) - \frac{1}{2} \bar{\nu}^2 (t - s) \right), \quad \bar{\nu} = \frac{\nu}{\sqrt{3}} \tag{33}
\]

To derive this approximation, we assume $\nu$ is small and start with a normal approximation

\[
\bar{\alpha}_t^2 = \alpha_s^2 \exp \left( \bar{\nu}(Z_t - Z_s) - \frac{1}{2} \bar{\nu}^2 (t - s) \right) \approx \quad \bar{\nu} = \frac{\nu}{\sqrt{3}} \tag{34}
\]

The $\zeta$ in (2) was defined in equation (A.57c) in [1]. When assuming CEV model for the forward process (i.e. $C(F) = F^\beta$), it has the form

\[
\zeta = \frac{\nu}{a} \int_K^f \frac{dF}{C(F)} = \frac{\nu}{a} \int_K^f \frac{du}{u^\beta} = \frac{\nu}{a} \cdot \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} = \frac{\nu}{a} \eta, \quad \eta = \frac{f^{1-\beta} - K^{1-\beta}}{1-\beta} \tag{35}
\]

By expanding, we have

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^y - e^{-y} = 2y \left( 1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots \right)
\]

\[
\Rightarrow p \left( e^{\frac{1-\beta}{2} q} - e^{-\frac{1-\beta}{2} q} \right) = p(1 - \beta)q \left( 1 + \frac{(1 - \beta)q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right),
\]

for $y = \frac{1 - \beta}{2} q$, $\quad p = \left( fK \right)^{\frac{1 - \beta}{2}}$, $\quad q = \ln \frac{f}{K}$ \tag{36}

\[
\Rightarrow f^\alpha - K^\alpha = (1 - \beta)pq \left( 1 + \frac{(1 - \beta)q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right)
\]

\[
\Rightarrow \eta = pq \left( 1 + \frac{(1 - \beta)q^2}{24} + \frac{(1 - \beta)^4 q^4}{1920} + \cdots \right)
\]
REFERENCES
