Risk Diversification in Portfolios

A final project report for the course

"Portfolio Theory and Applications"

Submitted

by

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ABSTRACT

This report summarizes Meucci's methodology [1] to analyze factors and sources of portfolio risk and therefore the risk diversification strategies. This methodology primarily focuses on analyzing the uncorrelated portfolio risk factors under various constraints for portfolio construction and its reallocation. Standard principal component analysis and conditional principal component analysis are performed to build the diversification distribution given the applied constraints. Effective number of bets, which is the exponential of the entropy of the diversification distribution, provides a powerful measure for the level of diversification in a portfolio. Finally the mean-diversification efficient frontier is introduced, which maximizes the diversification given an expected return.

TABLE OF CONTENTS

1. Introduction

Diversification is an important technique for reducing investment risk. It has been widely studied and applied to manage the systemic and non-systemic risks in portfolios. However the efficiency of diversification is heavily affected by correlations of underlying assets of portfolios. Careful analyses of the correlation structure and risk factors would lead to a deeper understanding of the risk sources and therefore a better diversification schemes and risk reallocation strategies.

Before looking into the theory, let's introduce a few simple definitions. Consider a portfolio consisting of N securities. The total return of the portfolio is defined as $R_w \equiv w^T R$, where the R is a column vector denoting the returns of the N securities within a given investment horizon and the vector w is the capital weights of each security in the portfolio. The risk of the portfolio is characterized by the variance of the total return, $Var(R_w) \equiv w^T \Sigma w$, where Σ is the variance-covariance matrix of security returns. In a fully uncorrelated markets, $Var(R_w) = \sum_{i=1}^{N} w_i{}^2 \cdot Var(R_i)$, which implies that maximum diversification can be achieved through equal variance-adjusted weights. The real markets are usually correlated. The risk of portfolio can be decomposed into parts and then explained by certain risk factors. Principal Component Analysis (PCA) is one of the common tools used to perform the risk decomposition and analyze the risk factors and contribution to the portfolio total risk. In a mathematical terminology, PCA of the risk matrix is equivalent to an orthogonal linear transformation that transforms the risk data to a new coordinate system such that the largest variance by any projection of the data comes to lie on the first coordinate, also known as the first principle component, and the second largest variance on the second, and so on. This report at first summarizes the PCA methodology and then discusses how this method can be applied to perform portfolio risk analyses and diversifications subject to various constraints.

2. Analysis of Portfolio Risk

2.1 Classical Principal Component Analysis

In modern portfolio theory, factor analysis is a very important concept. It basically states that the variability among the observed variables (e.g., prices of equities in a portfolio) can be described in terms of fewer unobserved variables known as factors. The observed variables are usually modeled as linear combinations of the common factors, in addition to an unaccounted error term. Factor analysis can be performed through, but not limited to, PCA method. Briefly speaking, PCA is a mathematical procedure performs, in our case, a variance-maximizing rotation of the variable space, which takes into account all variability in the variables. The idea is to transforms a set of possibly correlated random variables into a set of uncorrelated variables which are known as principal components. The first principal component accounts for as much of the variance in the data as possible, and each succeeding component accounts for as much of the remaining variance as possible.

The same idea applies to portfolio risks. The risk of portfolio can be explained through a set of uncorrelated factors. PCA is the process for factoring out these uncorrelated risk sources:

$$\Sigma \equiv E\Lambda E^T \Longrightarrow \Lambda \equiv E^T \Sigma E \tag{1}$$

where $\Lambda \equiv \mathrm{diag}(\lambda_1^2, \cdots, \lambda_N^2)$ and $E \equiv (e_1, \cdots, e_N)$ are the eigenvalue matrix and the eigenvector matrix, respectively. Entries on the main diagonal of Λ are eigenvalues in a descending order, and columns of E are the respective eigenvectors. Given that Σ is a symmetric positive definite matrix, the eigenvector matrix E is orthonormal, which performs purely a rotation transformation. The eigenvectors define a set of N uncorrelated portfolios, called principal portfolios, whose returns $\tilde{R} \equiv E^{-1}R$ are decreasingly responsible for the randomness in the market. The variances in returns of these uncorrelated principal portfolios are characterized by the corresponding eigenvalues. The weight vector w in terms of the underlying securities can be treated as a linear combination of the uncorrelated principal portfolios,

$$w \equiv E\widetilde{w} \Longrightarrow \widetilde{w} \equiv E^{-1}w = E^T w \tag{2}$$

Here we introduce the variance concentration curve as

$$\tilde{v}_n \equiv \widetilde{w}_n^2 \lambda_n^2\,, \quad n = 1, \cdots, N \tag{3}$$

and hence the total variance (i.e. risk) is

$$Var(R_w) \equiv w^T \Sigma w = w^T E \Lambda E^T w = \widetilde{w}^T \Lambda \widetilde{w} = \sum_1^N \widetilde{w}_n^2 \lambda_n^2 = \sum_1^N \tilde{v}_n \tag{4}$$

The entry $\tilde{v}_n$ represents the variance due to the n-th principal portfolio.

In terms of the variance concentration curve, we define two more terms. Firstly, the volatility concentration curve

$$s_n \equiv \frac{\widetilde{w}_n^2 \lambda_n^2}{Sd(R_w)} = \frac{\tilde{v}_n}{\sqrt{\sum_1^N \tilde{v}_n}}, \quad n = 1, \cdots, N \tag{5}$$

which represents a normalized decomposition of the variance concentration and describes the decomposition of volatility (or tracking error) into the contributions from each principal portfolio. Secondly, the diversification distribution,

$$p_n \equiv \frac{\widetilde{w}_n^2 \lambda_n^2}{Var(R_w)} = \frac{\tilde{v}_n}{\sum_1^N \tilde{v}_n}, \quad n = 1, \cdots, N \tag{6}$$

which can be treated not only as a normalized variance concentration curve, but also a vector of R-squares from regression of total portfolio return on the respective principal portfolios. In other words, the diversification distribution describes how much percentage of total variance can be explained by each of the principal portfolios.

Simply replacing the portfolio weight w with $w - b$, where b is a benchmark weight vector, the same analysis methodology can be applied to an active portfolio management against a benchmark. In this case, volatility concentration curve becomes tracking error concentration curve and diversification curve becomes the relative diversification distribution.

2.2 Conditional Principal Component Analysis

The conditional risk management arises from certain constraints, such that, for example, the risk of portfolio due to the market exposure cannot be diversified. If the market is the leading source of risk, the overall diversification level might be low, however by conditioning on this factor, the remaining risk of the portfolio may still be well diversified through analysis of the diversification distribution for the remaining factors. More in general, portfolios can be subject to a number of constraints. These constraints define a hyper-plane, a subset in the complete portfolio space where rebalancing is feasible (i.e. Rebalancing of a portfolio can only be performed on this hyper-plane). This constraint is defined through an implicit equation

$$A\Delta w \equiv 0 \tag{7}$$

where A is a conformable $K \times N$ matrix each row of which represents a constraint. For example, in the case of a budget constraint for portfolios, $\mathbf{1}^{\boldsymbol{T}} w \equiv 1$. The constraint is defined as $\mathbf{1}^{\boldsymbol{T}} \Delta w \equiv 0$ for rebalancing. Therefore the corresponding row in A is a vector of ones. It should be noted that the constraints defined through matrix A are imposed only on rebalancing, Δw. In other words, given an existing portfolio, these constraints define which rebalancing is allowed and which is not. Portfolio manager always wants to manage the risk of portfolio in a most efficient way, in order to minimize transaction costs and market impact. In this case, diversification distribution conditioning on these constraints provides a clear picture of the diversification structure of a portfolio when only specific rebalancing directions are allowed. The remarkable difference between the standard and the conditional portfolio management is that, given K constraints, rebalancing of conditional portfolio can only be achieved on the $(N - K)$ dimensional unconstrained sub-space. Thus the diversification distribution within the $(N - K)$ dimensional unconstrained sub-space is more interesting to the portfolio manager. Apparently, in this case, standard principal portfolios derived from unconditional case are no longer applicable to reflect the diversification distribution with constraints. We must decompose the total risk in terms of two parts: risk due to K constrained and risk due to $(N - K)$ unconstrained principal portfolios. This is equivalent to first finding unconstrained principal portfolios within the $(N - K)$ dimensional unconstrained sub-space conditioning on the K constraints, then finding the remaining constrained principal portfolios in the K dimensional constrained sub-space given the just-constructed unconstrained conditional principal portfolios.

In mathematics, this translates to a recursive definition:

$$
\begin{aligned}
& e_n \equiv \mathrm{argmax}_{\|e\|=1}\{e^T \Sigma e\} \\
& \qquad \text{subject to } \begin{cases} Ae \equiv 0 \\ e^T \Sigma e_j \equiv 0 \ \forall \text{ existing } e_j \end{cases} \\
& \qquad n = K + 1, \cdots, N \\
& \qquad j = K + 1, \cdots, n - 1
\end{aligned} \tag{8}
$$

where e_j's are the eigenvectors computed from preceding steps in the loop. For a general matrix A, the solution e_n, $n = K + 1, \cdots, N$ spans the $(N - K)$ dimensional unconstrained sub-space and represents principal portfolios that are mutually uncorrelated and decreasingly contribute to the total variance in the unconstrained sub-space. Once the $(N-K)$ unconstrained conditional principal portfolios are obtained, we follow another recursive procedure to construct the constrained principal portfolios:

$$\begin{aligned} & e_n \equiv \operatorname{argmax}_{\|e\|=1}\{e^T \Sigma e\} \\ & \qquad \text{subject to } e^T \Sigma e_j \equiv 0 \ \forall \text{ existing } e_j\,, \\ & \qquad n = 1, \cdots, K \\ & \qquad j = 1, \cdots, n-1 \quad \text{and} \quad j = K+1, \cdots, N \end{aligned} \tag{9}$$

As usual, these uncorrelated constrained principal portfolios decreasingly contribute to the total variance on the constrained K dimensional sub-space.

In more detail, the two recursive procedures described above can be stated in a more general algorithm:

$$\begin{aligned} & \hat{e} \equiv \operatorname{argmax}_{\|e\|=1}\{e^T \Sigma e\} \\ & \qquad \text{subject to } \ Be \equiv 0 \end{aligned} \tag{10}$$

where B is a conformable matrix. This is an equality constrained quadratic programming problem. In particular, for the first procedure defined by equation (8):

$$B \equiv \begin{pmatrix} A \\ e_{K+1}^T \Sigma \\ \vdots \\ e_{n-1}^T \Sigma \end{pmatrix} \tag{11}$$

and for the second procedure defined by equation (9), the vector e_n, $n = K+1, \cdots, N$ have been computed and known,

$$B \equiv \begin{pmatrix} e_1^T \Sigma \\ \vdots \\ e_{n-1}^T \Sigma \\ e_{K+1}^T \Sigma \\ \vdots \\ e_N^T \Sigma \end{pmatrix} \tag{12}$$

Note that the matrix B is growing as for every step the e_n computed from last step is aggregated into it.

To solve this problem, let's define the Lagrangian

$$\mathcal{L} \equiv e^T \Sigma e - \lambda(e^T e - 1) - \gamma^T B e \tag{13}$$

where scalar λ and vector γ are lagrangian multipliers. Its first derivative with respect to e ought to be zero, which gives

$$\frac{\partial \mathcal{L}}{\partial e} = 2\Sigma e - 2\lambda e - B^T \gamma = 0 \tag{14}$$

Multiply B to both sides, we have

$$\begin{aligned} 0 &= B \frac{\partial \mathcal{L}}{\partial e} = 2B\Sigma e - 2\lambda B e - BB^T \gamma \\ &= 2B\Sigma e - BB^T \gamma \end{aligned} \tag{15}$$

hence

$$\gamma = 2(BB^T)^{-1} B \Sigma e \tag{16}$$

And at last,

$$\begin{aligned} 0 &= \frac{1}{2} \frac{\partial \mathcal{L}}{\partial e} = \Sigma e - \lambda e - B^T (BB^T)^{-1} B \Sigma e \\ &= [I - B^T (BB^T)^{-1} B] \Sigma e - \lambda e \\ &= P \Sigma e - \lambda e \end{aligned} \tag{17}$$

where $P \equiv I - B^T(BB^T)^{-1}B$ and $\hat{e}$ is an eigenvector of $P\Sigma$. In other words, $\hat{e}$ is the solution of the problem $\hat{e} \equiv \mathrm{argmax}_{\|e\|=1}\{e^T P\Sigma e\}$, which is a quadratic programming problem without the explicit equality constraint $Be \equiv 0$.

Now define another problem

$$\tilde{e} \equiv \mathrm{argmax}_{\|e\|=1}\{e^T P\Sigma P^T e\} \tag{18}$$

Since $P\Sigma P^T$ is symmetric, the eigenvectors of $P\Sigma P^T$ are orthogonal. It is straightforward to verify that the matrix P is symmetric $P = P^T$ and idempotent $PP = P$. We have $P\Sigma P^T\tilde{e} = \lambda\tilde{e}$ and $P\Sigma P^T P\tilde{e} = PP\Sigma P^T\tilde{e} = \lambda P\tilde{e}$. Hence $P\tilde{e}$ is an eigenvector of $P\Sigma P^T$ and so is $\tilde{e}$. This implies $P\tilde{e} = \delta\tilde{e}$ for a suitable scalar δ. In addition, we have $P\Sigma P\tilde{e} = P\Sigma P^T\tilde{e} = P\Sigma P^T P\tilde{e} = \lambda P\tilde{e}$, so $P\tilde{e}$ is also an eigenvector of $P\Sigma$. This indicates that $\tilde{e}$ is an eigenvector of $P\Sigma$ and eventually $\hat{e} = \tilde{e}$. Instead of finding the eigenvector by matrix $P\Sigma$, we are in favor of the symmetric matrix $P\Sigma P^T$ for which the standard PCA method can be used.

By collecting the resulted conditional principal portfolios in a matrix $\hat{E} \equiv (\hat{e}_1, \cdots, \hat{e}_N)$, we can express the returns covariance in the same format as in the unconditional case,

$$\hat{E}\Sigma\hat{E}^T = \hat{\Lambda} \tag{19}$$

and the same analysis follows. The original weight w in portfolio can be treated as a linear combination of the conditional principal portfolios,

$$\tilde{\widehat{w}} \equiv \hat{E}^{-1}w \tag{20}$$

We then have the conditional variance concentration curve:

$$\tilde{\hat{v}}_n \equiv \tilde{\widehat{w}}_n^2\hat{\lambda}_n^2\,, \quad n = 1, \cdots, N \tag{21}$$

where the total portfolio variance is

$$Var(R_w) \equiv \sum_1^N \tilde{\hat{w}}_n^2 \hat{\lambda}_n^2 = \sum_1^N \tilde{\hat{v}}_n \tag{22}$$

The conditional volatility concentration curve is defined as:

$$\hat{s}_n \equiv \frac{\tilde{\hat{w}}_n^2 \hat{\lambda}_n^2}{Sd(R_w)} = \frac{\tilde{\hat{v}}_n}{\sqrt{\sum_1^N \tilde{\hat{v}}_n}}, \quad n = 1, \cdots, N \tag{23}$$

and finally the conditional diversification distribution is given by:

$$\hat{p}_n | A \equiv \frac{\tilde{\hat{v}}_n}{\sum_{K+1}^N \tilde{\hat{v}}_n}, \quad n = 1, \cdots, N \tag{24}$$

where A is the constraint matrix.

To perform portfolio management against a benchmark, the same strategy applies: simply replacing the portfolio weights with the vector of the relative bets ($w \mapsto w - b$) and eventually this will yield the relative conditional diversification distribution.

2.3 Entropy

The diversification distribution defined above can be treated as a set of probability masses, given that it is always defined, positive and sums to one, whether unconditional or conditional, absolute or relative to a benchmark. The level of diversification of a given portfolio can be viewed from the shape of the diversification distribution curve: a well diversified portfolio leads to approximately equal masses, total risk is uniformly distributed in all the principal portfolios; an ill diversified portfolio shows strong masses on some of the principal portfolios. To quantitatively measure the level of portfolio diversification, we introduce a term called entropy, as well as its exponential:

$$Ent \equiv -\sum_{K+1}^N p_n \ln p_n \quad \text{and} \quad \mathcal{N}_{Ent} \equiv \exp(Ent) \tag{25}$$

where K is the number of constraints (this reduces to unconditional case when $K = 0$). Given a portfolio consisting of N underlying securities, the value of $\mathcal{N}_{Ent}$ indicates the degree of aggregation of the total risk. When $\mathcal{N}_{Ent} = 1$, the portfolio risk is completely due to one single principal portfolio. When $\mathcal{N}_{Ent} = N$ (in unconditional case) or $\mathcal{N}_{Ent} = N - K$ (in conditional case), the portfolio risk uniformly spread among the N or $N - K$ available principal portfolios.

2.4 Mean Diversification Efficient Frontier

Mean-diversification efficient frontier is given by:

$$w_{\varphi} \equiv \operatorname{argmax}_{w \in C}\{\varphi \mu^T w + (1 - \varphi)\mathcal{N}_{Ent}(w)\} \tag{26}$$

where μ is the estimated expect returns and C is a set of investment constraints (including not only constraints on rebalancing directions but also constraints on structure of portfolios), parameter $\varphi \in [0,1]$ indicates the weight allocation on the diversification and on the expected return.

3. Analyses of Market Data

3.1 Datasets and Constraints

Standard and conditional risk analysis will be demonstrated based on two datasets:

1. The original dataset used in Meucci's paper [2], which gives:
    a. Σ : variance-covariance matrix of returns in 30 mid-cap stocks.
    b. $\mu \equiv 0.5\sigma$: expected returns, estimated by a risk-premium argument. σ is the vector of standard deviation of individual stock returns.
    c. w_b : benchmark in the same stocks with weights proportional to their market capitalization.
2. The data randomly selected from NYSE spreadsheet, which gives:

a. Σ : variance-covariance matrix of monthly returns in 30 random selected mid-cap stocks.

b. μ : expected returns, estimated by temporal average of individual stock monthly returns.

c. w_b : benchmark in the same stocks with weights proportional to their temporal average of market capitalization.

The diversification analyses are always performed against a benchmark w_b for generality. In the analyses, two types of constraints are involved. The following explains the constraints:

1. Portfolio constraints (define the ranges, structures of portfolio weights)
    a. Long-Short constraint: $-0.1 \leq w_i \leq 1.0, \quad i = 1, \cdots, N$
    b. Long-Only constraint: $0 \leq w_i \leq 1.0, \quad i = 1, \cdots, N$
    c. Budget constraint (always assumed): $\sum_{i=1}^{N} w_i = 1.0$
2. Rebalancing constraints (define the directions of re-allocation)
    a. Budget constraint: $\sum_{i=1}^{N} \Delta w_i = 0$ (no refinancing)
    b. No-trading constraint: $\Delta w_i = 0$ for $i = 1, 2, 3$ (first 3 stocks are suspended from trading, for example)

3.2 Analyses of Meucci's Data

3.2.1 Long-Short Portfolio with Unconstrained Rebalancing

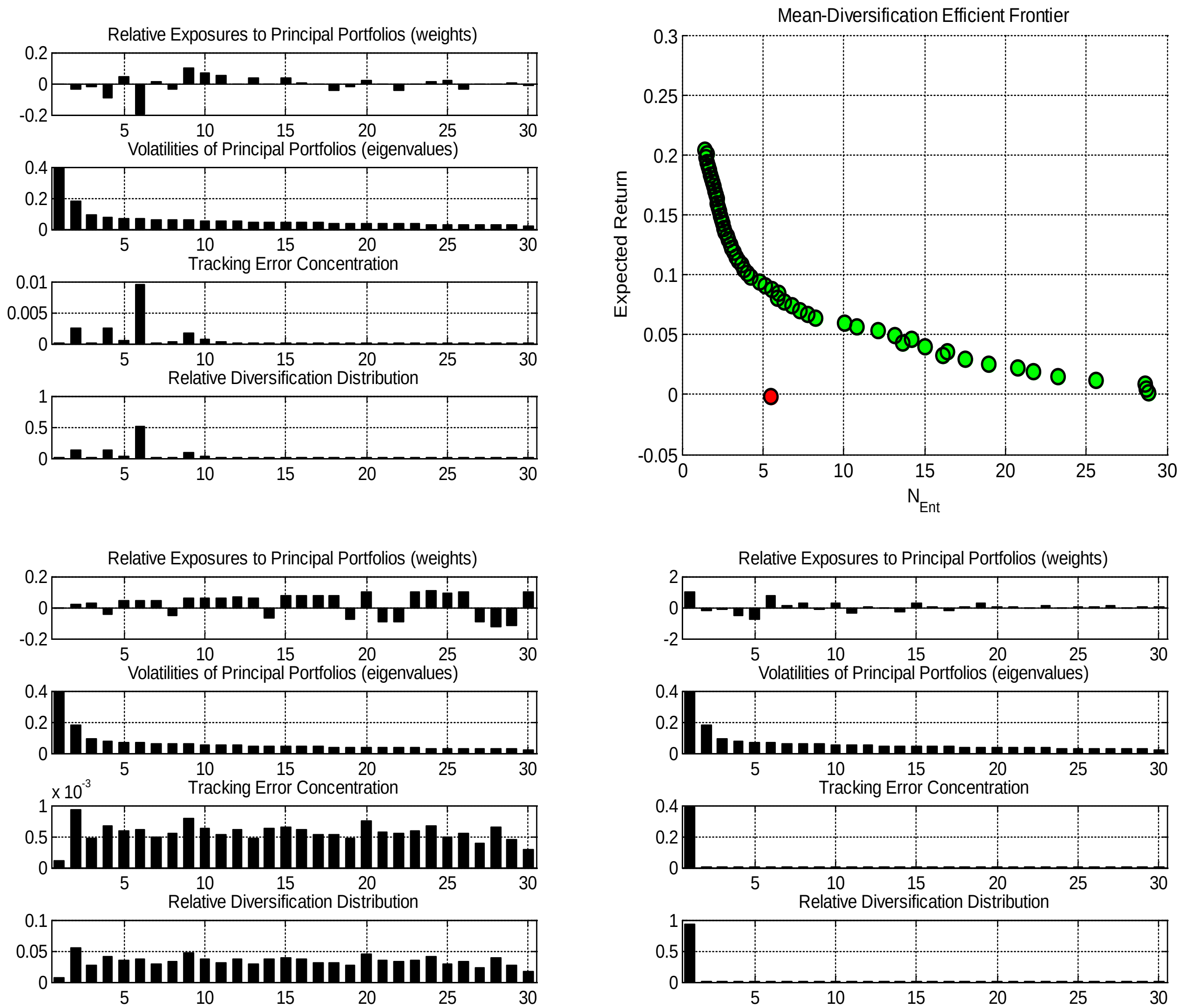

Figure 1. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\mathrm{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

In this case, portfolio follows the budget constraint and long-short constraint. Rebalancing is allowed in any directions. The mean-diversification efficient frontier is obtained through an optimization procedure. This basically reproduces the figures as show in Meucci's paper[1]. As shown in the Figure 1, total risk of the equally-weighted portfolio concentrates on the 6$^{\text{th}}$ principal portfolio. It ends up with $\mathcal{N}_{Ent} = 5.52$, and thus it is not well diversified and way off of the mean-diversification efficient frontier. The bottom two plots show the diversification profiles of the portfolios in two extreme cases. The portfolio with maximal $\mathcal{N}_{Ent}$ has a relative diversification distribution that nearly uniformly spreads among the principal portfolios, while the portfolio with the highest expect return almost fully concentrates on the first principal portfolio. As expected, lower diversification level leads to higher expected return, while higher diversification level leads to lower expected return.

3.2.2 Long-Only Portfolio with Unconstrained Rebalancing

In this case, portfolio follows the budget constraint and long-only constraint. Rebalancing is allowed in any directions. The results are shown in Figure 2 and look similar to that of unconstrained long-short portfolio except for the mean-diversification efficient frontier. Long-only portfolio has a more concave frontier. An increase in diversification reduces the expected return more rapidly than that of long-short portfolios. A possible explanation to this observation is that long-only portfolios cannot take advantage of long-short hedging and therefore has fewer ways to adjust the risk.

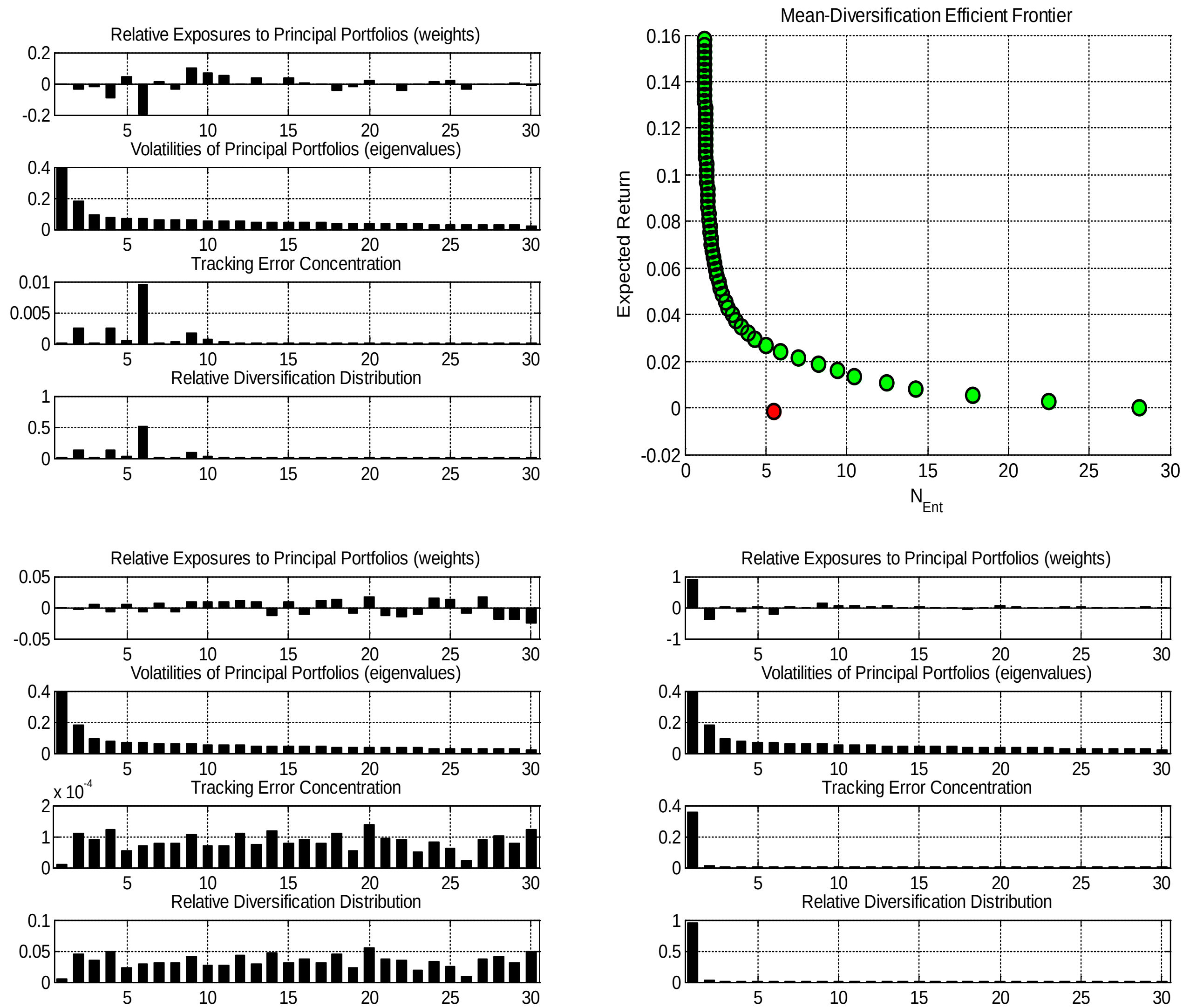

Figure 2. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\mathrm{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

3.2.3 Long-Only Portfolio with Constrained Rebalancing

In this case, portfolio must follow the budget constraint and the long-only constraint and rebalancing of portfolio must be subject to the budget constraint and the no-trading constraint. It should be noted that in the Figure 3, the first 4 principal portfolios are the constrained principal portfolios, while the remaining ones are unconstrained. Rebalancing can only be performed in terms of a linear combination of the unconstrained conditional principal portfolios.

It can be observed that the portfolio with maximal $\mathcal{N}_{Ent}$ has a relative diversification distribution uniformly spreads among the 26 unconstrained principal portfolios. The diversification significantly lowers the risk associated with the unconstrained principal portfolios, and eventually the risk due to the first constrained principal portfolio emerges as it cannot be diversified. This is consistent with the fact that portfolios have to bear the risks associated with the constrained principal portfolios.

Note that the portfolio with highest expect return almost fully concentrates on the first unconstrained principal portfolio (or the 5^{th} one, overall). This is because as the first unconstrained principal portfolio accounts for much of the total risk, it should also have the highest expected return.

Another observation is that the mean-diversification efficient frontier looks less smooth than in previous two cases. This is because the smoothness of the frontier curve heavily depends on the relative difference in the magnitude of eigenvalues. For example, if we have multiple identical eigenvalues, the total risk can be diversified into either of the corresponding principal portfolios or their linear combinations, which may leads to multiple optimal solutions. This is especially true for the first (largest) few eigenvalues, as their corresponding principal portfolios are the biggest factors accounting for the total risk. As a result, we will see jumps in the frontier curve due to the ambiguity. To illustrate this observation, we can compare the volatilities (i.e. eigenvalues) of principal portfolios between the unconstrained long-only case (Figure 2) and the constrained long-only case (Figure 3). We can see that the former has a smoother step-down in the largest volatilities, which leads to an overall bigger mutual difference among them and therefore improves the uniqueness of the optimal solution. The latter however has similar values for the 2nd, 3rd and 4th largest volatilities, which introduces ambiguities.

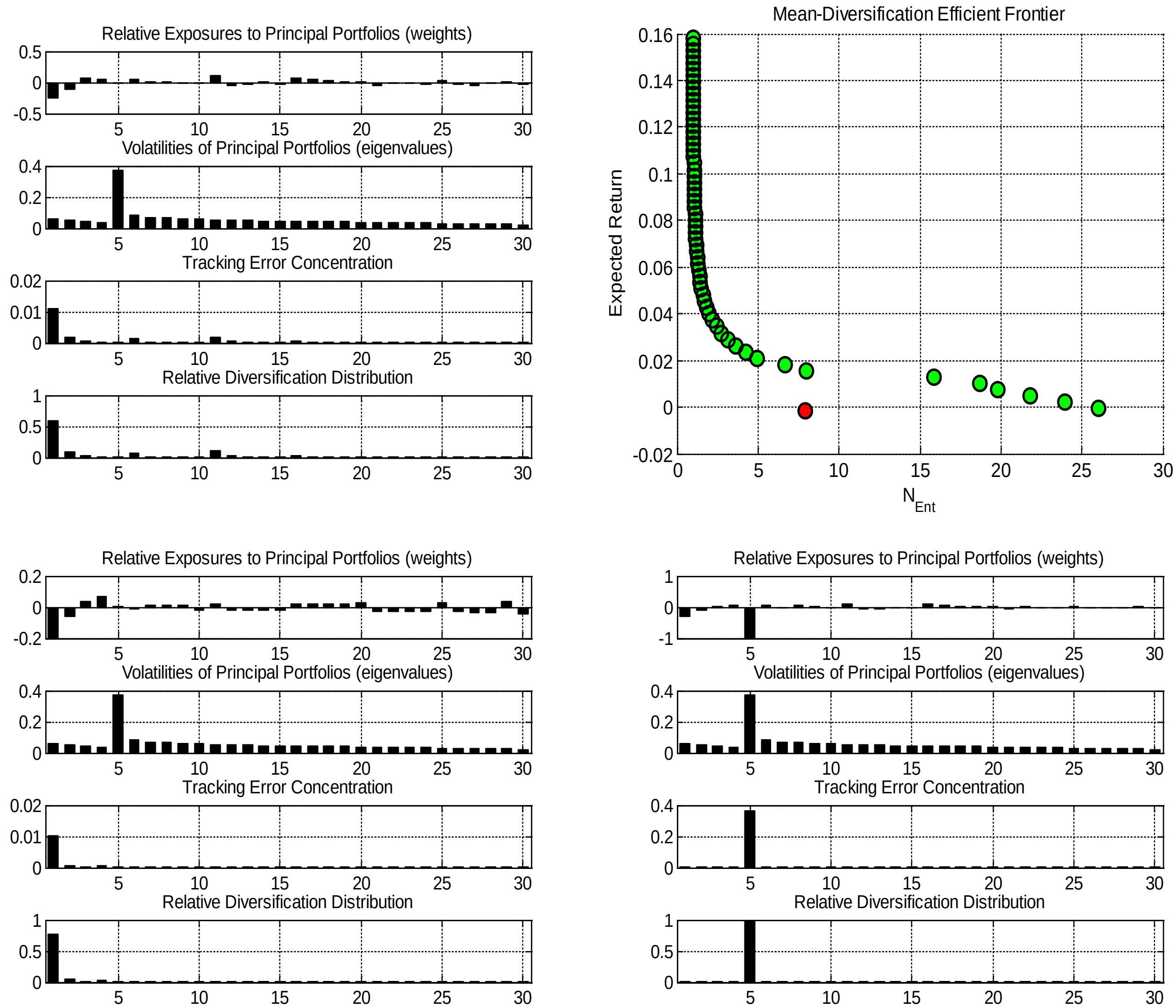

Figure 3. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\mathrm{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

3.2.4 Long-Short Portfolio with Constrained Rebalancing

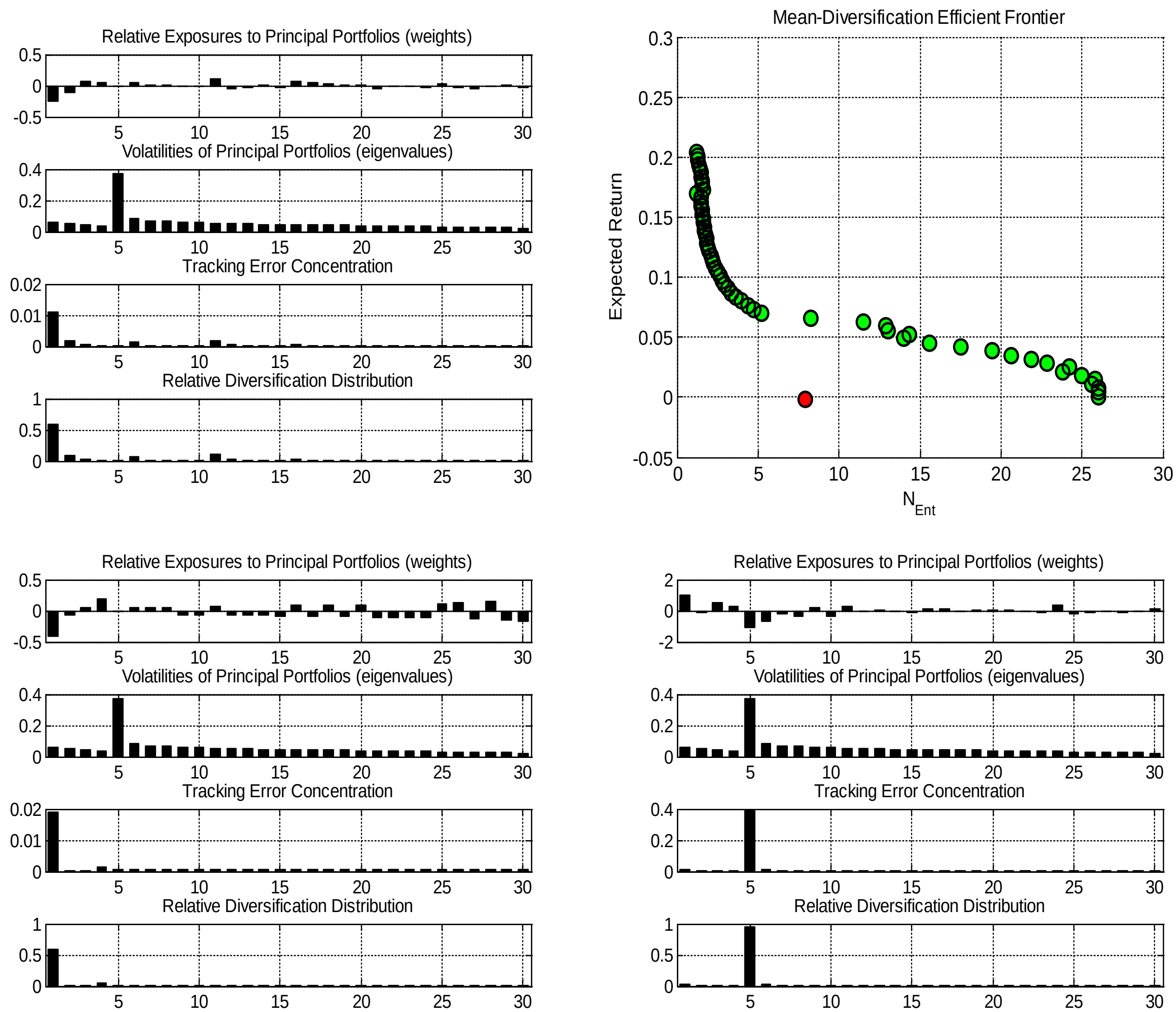

Figure 4. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\text{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

In this case, portfolio must obey the budget constraint and the long-short constraint. Rebalancing of portfolio must obey the budget constraint and the no-trading constraint. All the results are consistent with the logic that has been discussed in previous cases. The evident discontinuity in the mean-diversification efficient frontier is primarily due to the similarity in magnitudes among the 2nd, 3rd and 4th largest volatilities.

3.3 Analyses of NYSE Data

3.3.1 Long-Short Portfolio with Unconstrained Rebalancing

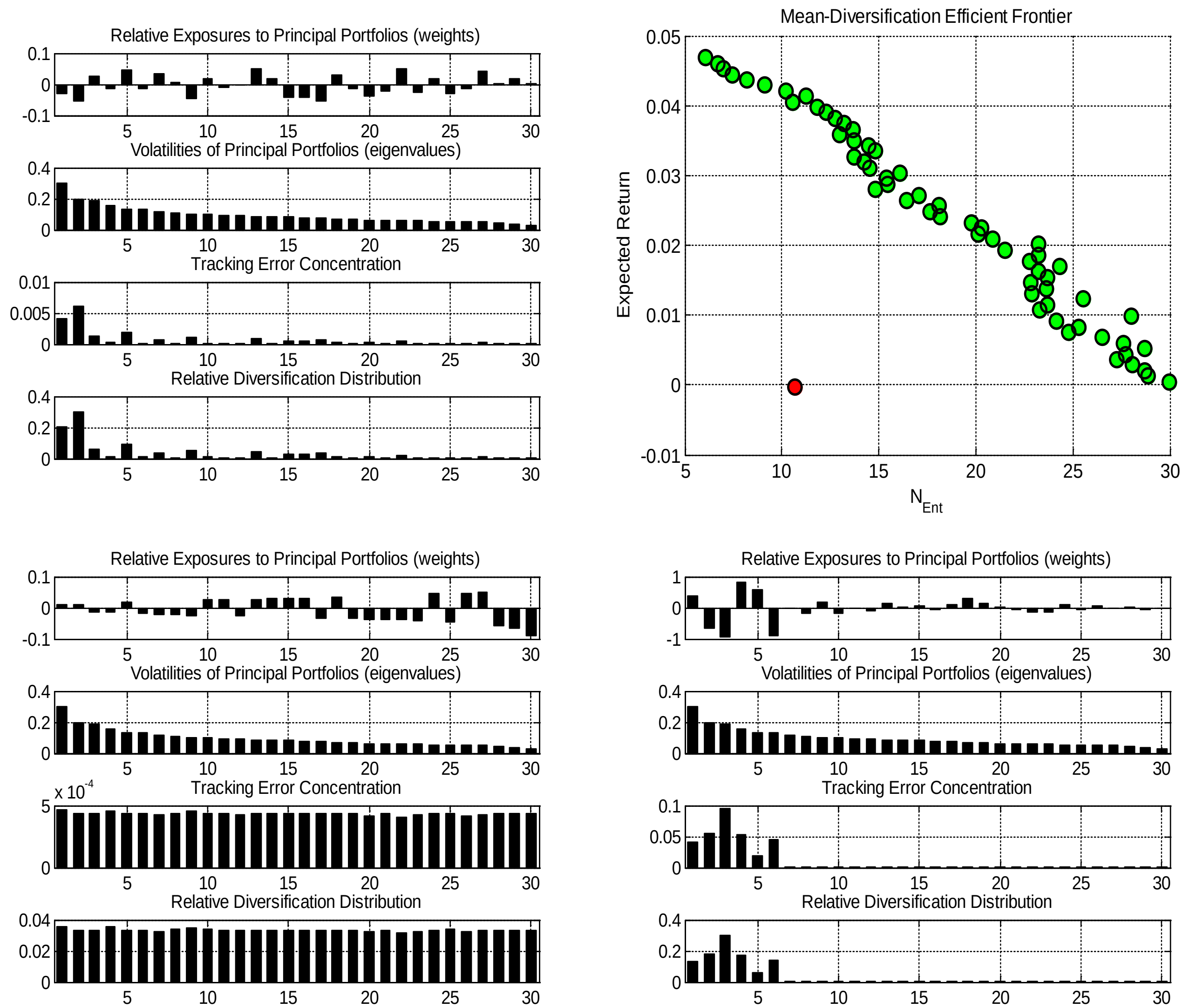

Figure 5. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\text{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

In this case, portfolio follows the budget constraint and long-short constraint. Rebalancing is not constrained. As one can see, the 2nd volatility is very close to the 3rd and the 5th is very close to the 6th. This is no doubt will introduce jumps in the mean-diversification efficient frontier curve as shown in the Figure 5. However the overall trend can still be clearly observed. Equally weighted portfolio is way off of the frontier curve and far suboptimal. Note that the risk of the portfolio with maximal expected return is strongly associated with multiple principal portfolios. This means the market is not one factor dominant and has multiple comparable uncorrelated risk sources.

3.3.2 Long-Only Portfolio with Unconstrained Rebalancing

In this case, portfolio follows the budget constraint and long-only constraint. Rebalancing is allowed in any directions. As expected, the results are consistent. One observation is that the overall expected return, as compared with the long-short unconstrained case, is quite small. Since this is a return against a benchmark, it can be reasonable.

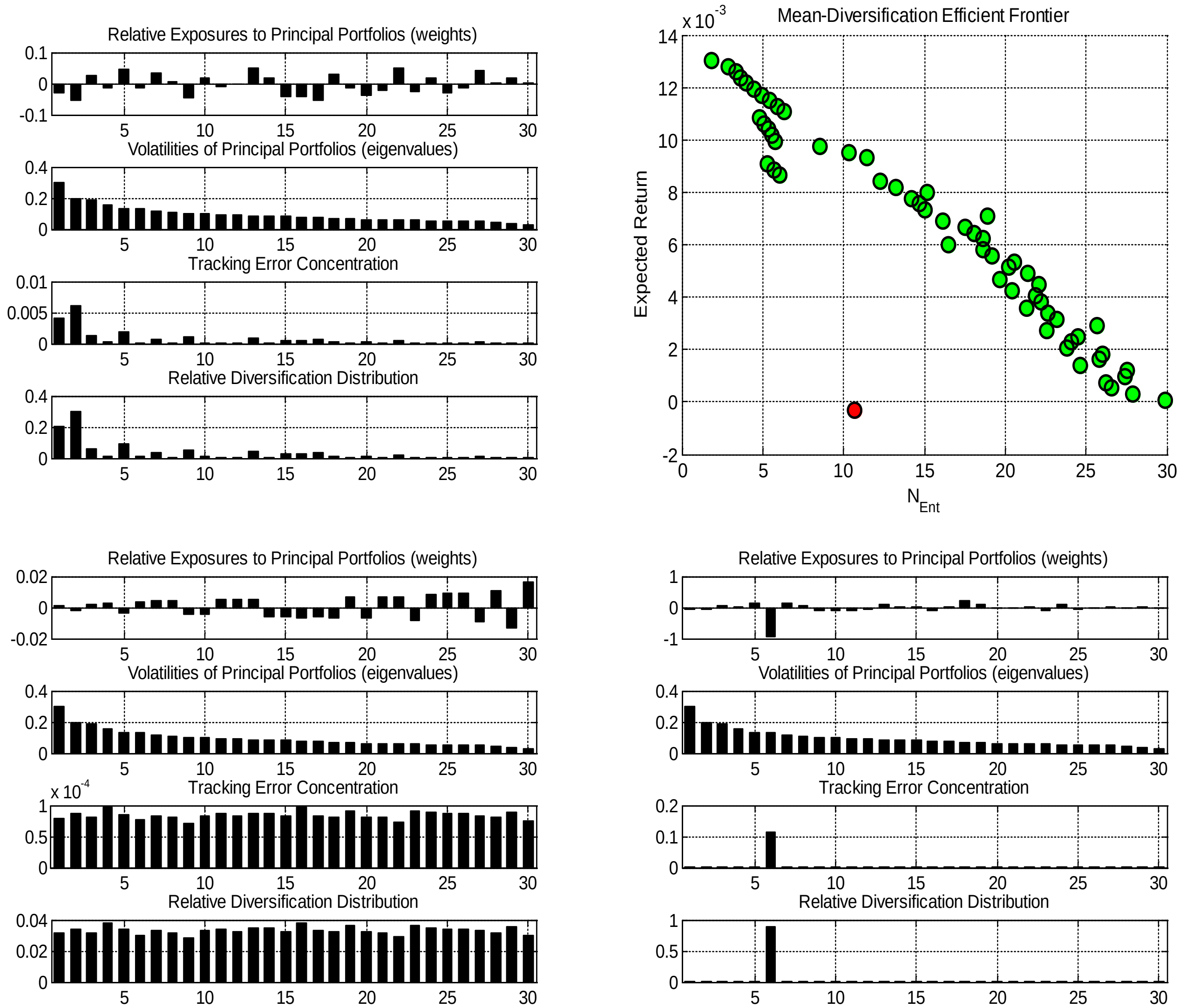

Figure 6. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\text{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

3.3.3 Long-Only Portfolio with Constrained Rebalancing

In this case, portfolio must follow the budget constraint and the long-only constraint and rebalancing of portfolio must follow the budget constraint and the no-trading constraint. As seen in the Figure 7, the 3 largest volatilities are approximately equal. This generates a great deal of ambiguities in searching for optimal solutions for mean diversification efficient frontier. As a result, the frontier curve looks lousy, however, the equally weighted portfolio is still clearly far suboptimal.

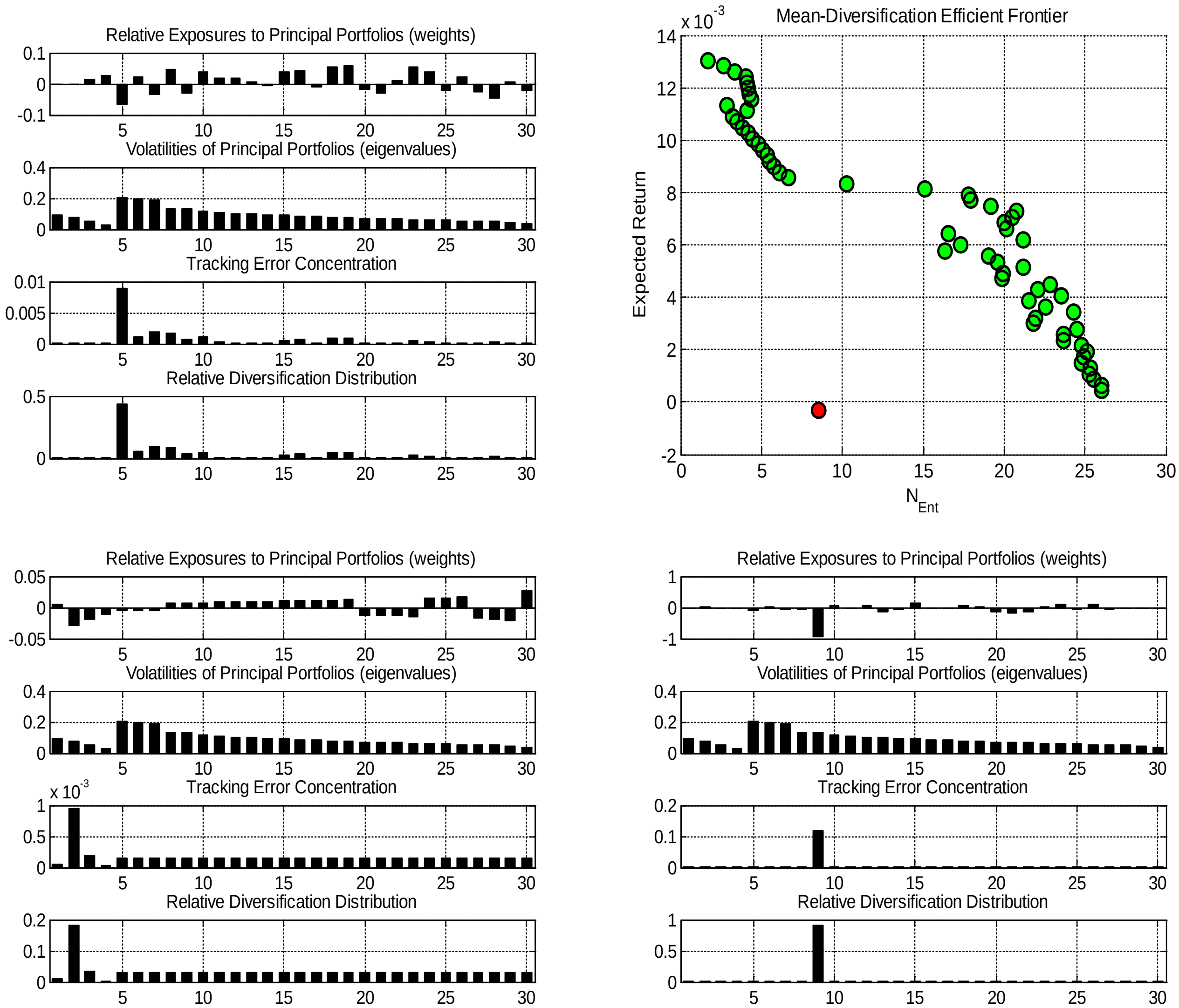

Figure 7. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\text{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

3.3.4 Long-Short Portfolio with Constrained Rebalancing

In this case, portfolio must obey the budget constraint and the long-short constraint and rebalancing of portfolio must obey the budget constraint and the no-trading constraint. All the results are consistent. As usual, for the portfolio with $\max(\mathcal{N}_{\text{Ent}})$, diversification relaxes much of the risks associated with unconstrained principal portfolios, while it can do little with the constrained principal portfolios.

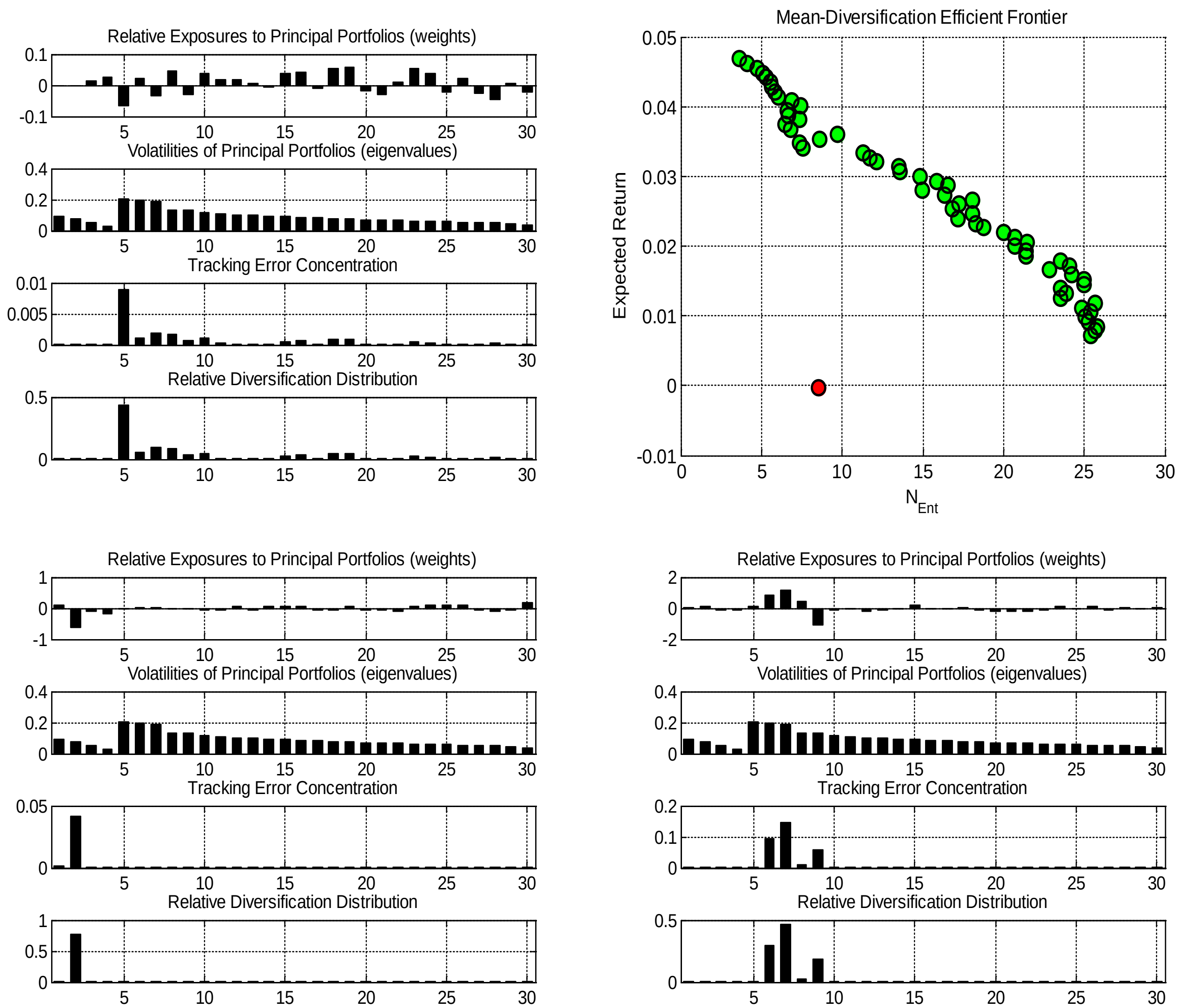

Figure 8. Top-left: equally weighted portfolio relative to benchmark. Top-right: mean-diversification efficient frontier (red dot represents the equally-weighted portfolio). Bottom-left: portfolio at $\max(\mathcal{N}_{\text{Ent}})$. Bottom-right: portfolio at $\max(\mu)$.

4. Conclusions

It has been demonstrated how to utilized the standard and the conditional principal component analysis method to decompose and analyze portfolio risks. Conditional principal component analysis method provides effective way to factorize the portfolio risk given constraints imposed on either portfolio structure or reallocation strategy. Diversification distribution, a direct product of this analysis, provides a powerful tool to analyze the fine structure of a portfolio concentration profile in various situations. The effective number of bets, defined based on the entropy of the diversification distribution, gives a quantitative index of diversification. And finally, the mean diversification efficient frontier provides a quantitative framework to manage the trade-off between the expected return and the effective number of uncorrelated bets.

REFERENCES

1. Meucci, A., “Managing Diversification”, *Risk*, 22, 5, 74-79 (2009)

2. Downloaded from *http://www.mathworks.com/matlabcentral/fileexchange/23271*

```matlab
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Project: Diversification
% Name: Changwei Xiong, 12/08/2009
%
%The program is based on Meucci's paper.
%The first dataset is Meucci's data used in
%his paper. The second dataset is adapted from Sakai
%as provided for the final project
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function Final
clc
close all
clear all

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% toggle the switches to choose the dataset and constraints,
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if 0 % select dataset
    load returns.csv
    load mktcaps.csv
    S = cov(returns);
    %mu = 0.5 * sqrt(diag(S));
    mu = mean(returns)';
    wb = 0.0*mean(mktcaps)'/sum(mean(mktcaps));
else
    load Data
    S = S;
    mu = Mu;
    wb = w_b;
end

% total number of assets
N = length(mu);
% equally weighted portfolio - benchmark
w0 = ones(N,1)/N - wb;


if 0 % select portfolio constraints
    % long-only constraint
    PCS.A = -eye(N); % A*w<=b ==> A*(w-wb)<=b-A*wb
    PCS.b = -PCS.A*wb;
    % budget constraint
    PCS.Aeq = ones(1,N); % Aeq*w = beq
    PCS.beq = 1 - PCS.Aeq*wb;
else
    % long-short constraint
    PCS.A = [eye(N); -eye(N)]; % A*w<=b ==> A*(w-wb)<=b-A*wb
    PCS.b = [ones(N,1); 0.1*ones(N,1)] - PCS.A*wb;
```

```
    % budget constrain
    PCS.Aeq = ones(1,N); % Aeq*w = beq
    PCS.beq = 1 - PCS.Aeq*wb;
end


if 0 % select rebalancing constraints
    % budget constraint and trading suspension for first 3 stocks
    RCS = [ones(1,N)];%; eye(3,N)];
else
    % no constraint
    RCS = [];
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

[E, L, G] = PrinComp(S, RCS);
MakePlots(E, L, w0);

w = MeanDivEffFrontier(G, mu, w0, PCS);
MakePlots(E, L, w(:,1));
MakePlots(E, L, w(:,end));
end

function w = Ptfl_at_MaxRet(mu, PCS)
% find w : max(mu'*w)|(A*x<=b & Aeq*x = beq)
w = linprog(-mu, PCS.A, PCS.b, PCS.Aeq, PCS.beq);
end

function w = Ptfl_at_MaxEnt(G, w0, PCS)
% find min of negative entropy == find max of entropy
w = fmincon(@NegativeEntropy, w0, ...
            PCS.A, PCS.b, PCS.Aeq, PCS.beq, ...
            [],[],[],...
            optimset('Algorithm','active-set', ...
                     'MaxFunEvals',10000, ...
                     'TolFun',1e-15, ...
                     'TolCon',1e-15, ...
                     'TolX'  ,1e-15));

    % embedded, compute negative entropy
    function negEnt = NegativeEntropy(w)
        v = G*w;
        varcon = v.*v;
        p = max(1e-10, varcon/sum(varcon));
        negEnt = p'*log(p);
    end
end

function w = MeanDivEffFrontier(G, mu, w0, PCS)
RmaxRet = mu' * Ptfl_at_MaxRet(mu, PCS);
RmaxEnt = mu' * Ptfl_at_MaxEnt(G, w0, PCS);
```

```

% mean-diversification efficient frontier
NP = 60;
ret = linspace(RmaxEnt, RmaxRet, NP);
%ret = ret(2:end-1);% remove the two extremas
%ret = linspace(RmaxEnt, 0.9*(RmaxRet-RmaxEnt)+RmaxEnt, NP);
NP = length(ret);

Nent = zeros(NP,1);
w = zeros(length(mu), NP);
for i = 1:NP
    pcs = PCS;
    pcs.Aeq=[pcs.Aeq; mu'];
    pcs.beq=[pcs.beq; ret(i)];
    w(:,i) = Ptfl_at_MaxEnt(G, w0, pcs);
    Nent(i) = EntropyExponential(G, w(:,i));
end

fontsize = 12;
linewidth = 2;
markersize = 10;
figure;
hold on;
plot(Nent, ret, 'o', ...
    'LineWidth',linewidth,...
    'MarkerEdgeColor','k',...
    'MarkerFaceColor','g',...
    'MarkerSize',markersize);
set(gca,'FontSize',fontsize);
ylabel('Expected Return','FontSize',fontsize);
xlabel('N_{Ent}','FontSize',fontsize);
title('Mean-Diversification Efficient Frontier','FontSize',fontsize);
grid on;

ret = mu'*w0;
Nent = EntropyExponential(G, w0)
plot(Nent, ret, 'o', ...
    'LineWidth',linewidth,...
    'MarkerEdgeColor','k',...
    'MarkerFaceColor','r',...
    'MarkerSize',markersize);
end

function Nent = EntropyExponential(G, w)
    s = G*w;
    p = max(1e-10, (s.*s)/(s'*s));
    Nent = exp(-p'*log(p));
end

function [E, L, G] = PrinComp(S, A)
if nargin==1 || isempty(A)
    [E, L] = eig(S);
```

```
    E = fliplr(E);
    L = flipud(diag(L));
    G = diag(sqrt(L))/E;
else
    [K, N] = size(A);
    E = [];
    B = A;
    for n = 1:N-K
        if n > 1
            B = [A; E'*S];
        end
        E = [E, getFirstPrinComp(S, B)];
    end

    for n = N-K+1:N
        B = E'*S;
        E = [E, getFirstPrinComp(S, B)];
    end
    % swap order
    E=[E(:,N-K+1:N) E(:,1:N-K)];
    L = diag(E'*S*E);
    [E, L] = SortPrinComp(E, L);
    G = diag(sqrt(L))/E;
    G = G(K+1:N, :);
end
end

%compute the first principal component
function e = getFirstPrinComp(S, B)
    P = eye(size(S,1));
    if rank(B)>0
        P = P - B'/(B*B')*B;
    end
    [E, L] = eig(P*S*P');
    [m, i]=max(diag(L));
    e=E(:,i);
end

function [E_, L_] = SortPrinComp(E, L)
% sort in descending order
L_ = sort(L, 'descend');
E_ = zeros(size(E,1));
for i = 1:length(L)
    E_(:,i) = E(:, L==L_(i));
end
end

function MakePlots(E, L, w)
width = 0.6;
fontsize = 11;

%[E, L] = SortPrinComp(E, L);
```

```
Xn = 1:size(E,1);

%relative exposures to the principal portfolios
figure;
hold on;
wt = E\w;
subplot(4,1,1);
bar(Xn, wt, width, 'k');
set(gca,'FontSize',fontsize);
xlim([min(Xn)-0.5, max(Xn)+0.5]);
title('Relative Exposures to Principal Portfolios (weights)','fontsize',fontsize);
grid on;

%Volatilities of principal portfolios
subplot(4,1,2);
bar(Xn, sqrt(L), width, 'k');
set(gca,'FontSize',fontsize);
xlim([min(Xn)-0.5, max(Xn)+0.5]);
title('Volatilities of Principal Portfolios (eigenvalues)','fontsize',fontsize);
grid on;

%variance concentration
V = wt.^2.*L;

var = sum(V);
std = sqrt(var);
%volatility/tracking error concentration curve
subplot(4,1,3);
bar(Xn, V/std, width, 'k');
set(gca,'FontSize',fontsize);
xlim([min(Xn)-0.5, max(Xn)+0.5]);
title('Tracking Error Concentration','fontsize',fontsize);
grid on;

subplot(4,1,4);
bar(Xn, V/var, width, 'k');
set(gca,'FontSize',fontsize);
xlim([min(Xn)-0.5, max(Xn)+0.5]);
%xlabel('X axis = Principal Portfolio Number','fontsize',fontsize);
title('Relative Diversification Distribution','fontsize',fontsize);
grid on;
end
```