Introduction to Interest Rate Models

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This note provides an introduction to interest rate models. At first, it attempts to explain the martingale pricing theory and change of numeraire technique in an intuitive way (hopefully!). Subsequently it covers several topics in rates models, including an introduction to rates market instruments, convexity adjustments, HJM framework, Quasi-Gaussian model, Linear Gaussian model, Hull-White 1-factor model, Jarrow-Yildirim model, and eventually the Libor Market model. Two main numerical method, PDE and Monte Carlo simulation, are also discussed.

It should be noted that nowadays the OTC derivative market has moved to central clearing. Majority of the OTC derivatives are now traded with collaterals to reduce the exposure of the counterparty credit risk. The classic single curve Libor discounting is no longer applicable. OIS discounting has been established as the new standard for collateral discounting in the OTC markets. In this notes, we will cover the Hull-White 1-factor model in multi-curve framework. However, some of the notes still rely on the classic Libor discounting, the formulas and equations derived in this way may appear a bit outdated but they do provide essential ingredients of development and applications of the models. Meanwhile, I am expecting to expand the notes to cover more in OIS discounting as well as other interesting topics in a long run.

1. Risk Neutral Measure

1.1. Heuristic Explanation: Risk Neutral Measure ⟺ No Arbitrage

Suppose there is a *non-dividend bearing* and *market tradable* asset A_t whose spot price follows a stochastic differential equation (SDE)

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t \tag{1}$$

The solution to this equation is

$$A_T = A_s \exp\left(\mu(T-s) - \frac{1}{2}\sigma^2(T-s) + \sigma(W_T - W_s)\right) \tag{2}$$

where $(W_T - W_s) \sim \mathcal{N}(0, T-s)$ is a normally distributed random variable with mean 0 and variance $(T-s)$. Here we denote t the time variable, μ a constant drift and σ a constant volatility. Unless otherwise stated, we always assume $s < t < T$ for start s and maturity T. The extra term $-\frac{1}{2}\sigma^2(T-s)$ in the exponent comes from the fact that $f(x) = \exp(x)$ is a convex function.

Financial Derivatives are typically priced assuming no frictions in financial markets. Under this assumption, one can find a portfolio strategy that does not use the derivative and only requires an initial investment such that the portfolio pays the same as the derivative at maturity. The portfolio is called a replicating portfolio. The derivative must be worth the same as the replicating portfolio if financial markets are frictionless, otherwise there will be an opportunity to make a risk-free profit (i.e., arbitrage). Suppose that at time t we long a forward contract that will pay 0 cash (i.e., contractual price = 0) for one unit of the asset upon maturity at a future time $T > t$. The payoff cashflow at T is just the terminal spot price A_T. Since A_T is a random variable at time t, we may assume that current value of the forward contract can be expressed as an expectation of the discounted contingent claim under a certain probability measure $(*)$. That is $F_{t,T} = \mathbb{E}^*\left[A_T \frac{D_T}{D_t}\middle| \mathcal{F}_t\right] = \mathbb{E}_t^*\left[A_T \frac{D_T}{D_t}\right]$, where the expectation $\mathbb{E}[\cdot|\mathcal{F}_t]$ conditional on filtration $\mathcal{F}_t$ is denoted by $\mathbb{E}_t[\cdot]$ for brevity. The discount factor D_t can also be a random variable, which is defined as

$$D_t \equiv \exp\left(-\int_s^t r_u du\right) \qquad \text{or} \qquad dD_t = -r_t D_t dt \tag{3}$$

with instantaneous risk free rate r_t. To replicate the derivative payoff cashflow, one can (statically) hold one unit of asset A_t at time t (plus 0 amount of money market account $M_t = D_t^{-1}$). Upon maturity, this will have the same value as the forward contract (i.e., the same payoff) regardless of the stochasticity of the asset price. To avoid an arbitrage opportunity, current value of the forward contract must equal to the present spot price of the asset, that is $F_{t,T} = \mathbb{E}_t^*\left[A_T \frac{D_T}{D_t}\right] = A_t$. However, a question arises: “Under which probability measure does the equality hold?”

Notice that D_t is known at t and can be moved out of the expectation, we therefore have $\mathbb{E}_t^*[A_T D_T] = A_t D_t$, which indicates that under such a probability measure the discounted spot process is a martingale. Let us first assume the expectation is under physical measure $\mathbb{P}$, we then have the dynamics of the discounted spot process as

$$\frac{d(A_t D_t)}{A_t D_t} = \frac{dD_t}{D_t} + \frac{dA_t}{A_t} + \frac{dA_t dD_t}{A_t D_t} = (\mu - r)dt + \sigma dW_t \tag{4}$$

The integrated solution to the above SDE is

$$A_T D_T = A_t D_t \exp\left((\mu - r)(T - t) - \frac{1}{2}\sigma^2(T - t) + \sigma(W_T - W_t)\right) \tag{5}$$

and hence

$$\mathbb{E}_t[A_T D_T] = A_t D_t e^{(\mu-r)(T-t)} \underbrace{\mathbb{E}_t\left[\exp\left(-\frac{1}{2}\sigma^2(T - t) + \sigma(W_T - W_t)\right)\right]}_{=1} = A_t D_t e^{(\mu-r)(T-t)} \tag{6}$$

It is obvious that $\mathbb{E}_t[A_T D_T] \neq A_t D_t$ unless $\mu = r$. To satisfy the arbitrage free condition $F_{t,T} = A_t$, we must find an equivalent probability measure under which $\mathbb{E}_t^*[A_T D_T] = A_t D_t$ is a martingale (i.e., the equivalent martingale measure). In other words, under such a measure, we must have $\mu = r$. Heuristically speaking, under this measure one has no preference on risky or riskless underlying assets. All assets would have a unique rate of return at r_t. This is known as *risk neutral measure*, denoted commonly by $\mathbb{Q}$.

Let λ be the market price of risk (i.e., the risk premium that investor demand to bear risk), such that $\mu - r = \lambda\sigma$. The asset price process in (1) becomes

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW = r dt + \sigma d\widetilde{W}_t \tag{7}$$

for $d\widetilde{W}_t = dW_t + \lambda dt$ being a Brownian motion under $\mathbb{Q}$. Hence the discount asset $d(A_t D_t) = A_t D_t \sigma d\widetilde{W}_t$ becomes a martingale and thus $\widetilde{\mathbb{E}}_t[A_T D_T] = A_t D_t$. (We usually use a "tilde" mark to denote quantities associated with risk neutral measure. For example, the $\widetilde{\mathbb{E}}_t[\cdot]$ denotes an expectation under risk neutral measure $\mathbb{Q}$.)

Based on the heuristic explanation, we can generalize the risk neutral pricing theory as follows: one agent can choose an initial capital X_t (e.g., in our example the value of A_t) and a portfolio strategy Δ_t (e.g., long one unit of asset A) to hedge a short position in a derivative (e.g., the forward contract that pays one unit of asset A upon maturity) whose payoff upon maturity is V_T (e.g., A_T in our example). Namely, we want to have $X_T = V_T$ almost surely. The value X_t of the hedging portfolio is the capital needed at time t in order to successfully complete the hedge. Hence, we call this value the price of the derivative V_t at time t (otherwise arbitrage arises). This arbitrage free property gives rise to the classic risk neutral pricing formula (also known as martingale pricing formula)

$$V_t D_t = \widetilde{\mathbb{E}}_t[V_T D_T] \;\; \forall \;\; t \le T \tag{8}$$

1.2. Equivalent Probability Measure and Girsanov Theorem

Two probability measures $\mathbb{N}$ and $\mathbb{U}$ on $(\Omega, \mathcal{F})$ are said to be *equivalent* if they agree which sets in $\mathcal{F}$ have probability zero. Let $(\Omega, \mathcal{F}, \mathbb{N})$ be a probability space and let Z be an almost surely non-negative random variable with $\mathbb{E}^{\mathbb{N}}[Z] = 1$. For $A \in \mathcal{F}$, define

$$\mathbb{U}(A) = \int_A Z(\omega)\, d\mathbb{N}(\omega) \tag{9}$$

then $\mathbb{U}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\mathbb{E}^{\mathbb{U}}[X] = \mathbb{E}^{\mathbb{N}}[XZ] \tag{10}$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}^{\mathbb{N}}[X] = \mathbb{E}^{\mathbb{U}}\left[\frac{X}{Z}\right] \tag{11}$$

The Z is called the *Radon-Nikodym derivative* of $\mathbb{U}$ with respect to $\mathbb{N}$ and we write

$$Z \equiv \frac{d\mathbb{U}}{d\mathbb{N}} \tag{12}$$

This indicates that Z is like a likelihood ratio of the two probability measures.

Lastly, we introduce the *Girsanov Theorem* [1], which describes how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure. The theorem is especially important and has profound influence on the theory of financial mathematics. To make it more general, here we focus on a multi-dimensional version of the theorem (which can be easily reduced to 1D): Let $t > s$ be a fixed positive time and let θ_t be an adapted n-dimensional process. Also Let $W_t^{\mathbb{N}}$ be an n-dimensional Brownian motion under the measure $\mathbb{N}$ (with correlated components, e.g., $(dW_t^{\mathbb{N}})(dW_t^{\mathbb{N}})' = \rho dt$ where matrix ρ is the instantaneous correlation and the prime symbol denotes a matrix transpose). If we define

$$Z_t = \exp\left(-\frac{1}{2}\int_s^t \theta_u' \rho \theta_u du - \int_s^t \theta_u' dW_u^{\mathbb{N}}\right) \quad \text{and} \quad dW_t^{\mathbb{U}} = dW_t^{\mathbb{N}} + \rho\theta_t dt \tag{13}$$

then under the measure $\mathbb{U}$ given by

$$\mathbb{U}(A) = \int_A Z(\omega)\, d\mathbb{N}(\omega), \qquad \forall\, A \in \mathcal{F} \tag{14}$$

the process $W_t^{\mathbb{U}}$ is an n-dimensional Brownian motion (with instantaneous correlation ρ).

One example to show the claim is that, if under $\mathbb{U}$ we have a martingale process for $s < t$

$$X_t = X_s \exp\left(-\frac{1}{2}\int_s^t \mathbb{1}' \sigma_u \rho \sigma_u \mathbb{1} du + \int_s^t \mathbb{1}' \sigma_u dW_u^{\mathbb{U}}\right) \tag{15}$$

where $\sigma_u$is a diagonal matrix[1] representing an adapted volatility process, then according to (13) we have

$$X_t = X_s \exp\left(-\frac{1}{2}\int_s^t \mathbb{1}' \sigma_u \rho \sigma_u \mathbb{1} du + \int_s^t \mathbb{1}' \sigma_u dW_u^{\mathbb{N}} + \int_s^t \mathbb{1}' \sigma_u \rho \theta_u du\right) \quad \text{and}$$

$$X_t Z_t = X_s \exp\left(-\frac{1}{2}\int_s^t \mathbb{1}' \sigma_u \rho \sigma_u \mathbb{1} du + \int_s^t \mathbb{1}' \sigma_u \rho \theta_u du - \frac{1}{2}\int_s^t \theta_u' \rho \theta_u du + \int_s^t (\sigma_u \mathbb{1} - \theta_u)' dW_u^{\mathbb{N}}\right) \tag{16}$$

$$= X_s \exp\left(-\frac{1}{2}\int_s^t (\sigma_u \mathbb{1} - \theta_u)' \rho (\sigma_u \mathbb{1} - \theta_u) du + \int_s^t (\sigma_u \mathbb{1} - \theta_u)' dW_u^{\mathbb{N}}\right)$$

The $X_t Z_t$ process is a martingale under $\mathbb{N}$. Hence the relationship in (10) is satisfied

$$\mathbb{E}_s^{\mathbb{U}}[X_t] = \mathbb{E}_s^{\mathbb{N}}[X_t Z_t] = X_s \tag{17}$$

[1] To be more flexible, we write the σ_u here as a diagonal matrix rather than a column vector.

In fact, the market price of risk process can be regarded as a *Radon-Nikodym derivative*. It allows to alter the drift of a Brownian motion W_t under physical measure, to create a new Brownian motion $\widetilde{W}_t$ under risk neutral measure.

1.3. Change of Numeraire

A numeraire is any positive *non-dividend bearing* and *market tradable* asset. Intuitively, a numeraire is a reference asset that is chosen so as to normalize all other asset prices with respect to it. We are interested in the change of numeraires because changing between probability measures is usually associated with numeraire changes. To explain this, let us first define two market tradable spot processes

$$\begin{gathered}\frac{dS}{S} = \mu_S dt + \mathbb{1}'\sigma_S dW_t = rdt + \mathbb{1}'\sigma_S d\widetilde{W}_t, \qquad \frac{dN}{N} = \mu_N dt + \mathbb{1}'\sigma_N dW_t = rdt + \mathbb{1}'\sigma_N d\widetilde{W}_t \\ \text{and} \quad \underset{n\times 1}{\mathbb{1}} = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{dW_t} = \begin{bmatrix} \vdots \\ dW_{i;t} \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{dW_t}\underset{1\times n}{dW_t'} = \underset{n\times n}{\rho}\, dt\end{gathered} \tag{18}$$

where we assume their price dynamics is driven by an common n-dimensional Brownian motion dW_t under physical measure (or by $d\widetilde{W}_t$ under risk neutral measure). The $\mathbb{1}$ is a column vector of 1's used to aggregate vector/matrix elements. The σ_S and σ_N are two $n \times n$ diagonal matrices denoting two adapted volatility processes associated with asset S and N respectively. Note that the components of dW_t may be correlated by matrix ρ, e.g., $dW_t dW_t' = \rho dt$. Let λ be an $n \times 1$ vector of the market price of risk. The arbitrage-free condition claims that we must have a unique λ such that

$$\mu_S - r = \mathbb{1}'\sigma_S\lambda \qquad \text{and} \qquad \mu_N - r = \mathbb{1}'\sigma_N\lambda \tag{19}$$

With this λ, we can write $d\widetilde{W}_t = dW_t + \lambda dt$.

Choosing a numeraire N implies that the relative price S/N is considered, instead of the asset price S alone. The following describes the dynamics of S relative to the numeraire N

$$\begin{aligned}&d\frac{S}{N} = Sd\frac{1}{N} + \frac{dS}{N} + dSd\frac{1}{N} \\ &= \frac{S}{N}(-\mu_N dt - \mathbb{1}'\sigma_N dW_t + \mathbb{1}'\sigma_N\rho\sigma_N\mathbb{1}dt) + \frac{S}{N}(\mu_S dt + \mathbb{1}'\sigma_S dW_t) - \frac{S}{N}\mathbb{1}'\sigma_S\rho\sigma_N\mathbb{1}dt\end{aligned} \tag{20}$$

where the numeraire inverse has the dynamics

$$d\frac{1}{N} = -\frac{1}{N^2}dN + \frac{1}{2}\frac{2}{N^3}dNdN = \frac{1}{N}\left(-\mu_N dt - \mathbb{1}'\sigma_N dW_t + \mathbb{1}'\sigma_N\rho\sigma_N\mathbb{1}dt\right) \tag{21}$$

Collecting the terms and using the relations in (19), we have

$$\begin{aligned}
\frac{d\frac{S}{N}}{\frac{S}{N}} &= \underbrace{-\mathbb{1}'\sigma_S\rho\sigma_N\mathbb{1}dt}_{\text{covariance}} + \underbrace{\mathbb{1}'\sigma_N\rho\sigma_N\mathbb{1}dt - \mu_N dt - \mathbb{1}'\sigma_N dW_t}_{\text{due to } 1/N} + \underbrace{\mu_S dt + \mathbb{1}'\sigma_S dW_t}_{\text{due to } S} \\
&= (\mu_S - \mu_N)dt + \mathbb{1}'(\sigma_S - \sigma_N)dW_t - \mathbb{1}'(\sigma_S - \sigma_N)\rho\sigma_N dt \\
&= \mathbb{1}'(\sigma_S - \sigma_N)\lambda dt + \mathbb{1}'(\sigma_S - \sigma_N)dW_t - \mathbb{1}'(\sigma_S - \sigma_N)\rho\sigma_N\mathbb{1}dt \\
&= \underbrace{\mathbb{1}'(\sigma_S - \sigma_N)(dW_t + \lambda dt - \rho\sigma_N\mathbb{1}dt)}_{\text{Under Physical Measure } \mathbb{P}} = \underbrace{\mathbb{1}'(\sigma_S - \sigma_N)\left(d\widetilde{W}_t - \rho\sigma_N\mathbb{1}dt\right)}_{\text{Under Risk Neutral Measure } \mathbb{Q}} = \underbrace{\mathbb{1}'(\sigma_S - \sigma_N)dW_t^{\mathbb{N}}}_{\text{Under Measure } \mathbb{N}}
\end{aligned} \tag{22}$$

where the probability measure $\mathbb{N}$ is associated with the numeraire N. It tells that the Brownian motions under different probability measures have the following relationships

$$\underbrace{dW_t + \lambda dt}_{\text{Under } \mathbb{P}} = \underbrace{d\widetilde{W}_t}_{\text{Under } \mathbb{Q}} = \underbrace{dW_t^{\mathbb{N}} + \rho\sigma_N\mathbb{1}dt}_{\text{Under } \mathbb{N}} \tag{23}$$

The (22) shows the process S/N is a martingale under $\mathbb{N}$, which by definition of martingale implies the following fundamental relationship

$$\frac{S_t}{N_t} = \mathbb{E}_t^{\mathbb{N}}\left[\frac{S_T}{N_T}\right] \tag{24}$$

In fact, the aforementioned risk neutral pricing formula (8) is just a special case of (24), where the money market account M_t is used as the numeraire (recall that $D_t = M_t^{-1}$).

Moreover, if we define another probability measure $\mathbb{U}$ associated with numeraire U, then given the start s the Radon-Nikodym derivative that forms the measure $\mathbb{U}$ from measure $\mathbb{N}$ reads

$$Z_t = \frac{d\mathbb{U}}{d\mathbb{N}} = \frac{U_t}{U_s}\left(\frac{N_t}{N_s}\right)^{-1} = \frac{U_t N_s}{U_s N_t} \tag{25}$$

According to (22), we have

$$d\frac{U}{N} = \frac{U}{N}\mathbb{1}'(\sigma_U - \sigma_N)dW_t^{\mathbb{N}} \tag{26}$$

and its integrated solution is

$$\frac{U_t}{N_t} = \frac{U_s}{N_s}\exp\left(-\int_s^t \mathbb{1}'(\sigma_N - \sigma_U)dW_u^{\mathbb{N}} - \frac{1}{2}\int_s^t \mathbb{1}'(\sigma_N - \sigma_U)\rho(\sigma_N - \sigma_U)\mathbb{1}du\right) \tag{27}$$

Here we follow the Girsanov Theorem to define

$$Z_t = \frac{U_t N_s}{U_s N_t} = \exp\left(-\int_s^t \mathbb{1}'(\sigma_N - \sigma_U)dW_u^{\mathbb{N}} - \frac{1}{2}\int_s^t \mathbb{1}'(\sigma_N - \sigma_U)\rho(\sigma_N - \sigma_U)\mathbb{1}du\right) \tag{28}$$

hence the resulted Brownian motion under measure $\mathbb{U}$ is

$$W_t^{\mathbb{U}} = W_t^{\mathbb{N}} + \int_s^t \rho(\sigma_N - \sigma_U)\mathbb{1}du \tag{29}$$

This is consistent with (23) in differential form

$$dW_t^{\mathbb{U}} + \rho\sigma_U\mathbb{1}dt = dW_t^{\mathbb{N}} + \rho\sigma_N\mathbb{1}dt = d\widetilde{W}_t \tag{30}$$

Suppose that an asset price process X follows an SDE under measure $\mathbb{N}$ and another SDE under measure $\mathbb{U}$, respectively, we must have

$$\frac{dX}{X} = \mu_X^{\mathbb{N}}dt + \mathbb{1}'\sigma_X dW_t^{\mathbb{N}} = \mu_X^{\mathbb{U}}dt + \mathbb{1}'\sigma_X dW_t^{\mathbb{U}} \tag{31}$$

Then the adjustment in drift term due to the change of numeraire from N to U is

$$\mu_X^{\mathbb{U}} = \mu_X^{\mathbb{N}} + \mathbb{1}'\sigma_X\left(dW_t^{\mathbb{N}} - dW_t^{\mathbb{U}}\right) \quad \Longrightarrow \quad \mu_X^{\mathbb{U}} = \mu_X^{\mathbb{N}} - \mathbb{1}'\sigma_X\rho(\sigma_N - \sigma_U)dt \tag{32}$$

1.4. Forward vs. Futures

Now we will use an example to show the application of the change of numeraire, and eventually explain the difference in values of Forward and Futures contracts.

Forward Contract: Suppose at time t one may enter into a forward contract on an underlying S_t with a quoted strike price K_t (i.e., the forward price) at zero cost $V_t = 0$, and the contract position may not be closed out until it matures at T. Upon delivery, the contract settles the spot-strike difference $(S_T - K_t)$ in cash. There is only one cashflow comes solely from the end. If we define the value of the payoff cashflow as $V_T = S_T - K_t$, we can use the risk neutral pricing formula to derive the strike price K_t that makes $V_t = 0$

$$
\begin{aligned}
&0 = V_t = \frac{1}{D_t}\widetilde{\mathbb{E}}_t[D_T V_T] = \frac{1}{D_t}\widetilde{\mathbb{E}}_t[D_T S_T] - \frac{K_t}{D_t}\widetilde{\mathbb{E}}_t\left[D_T P_{T,T}\right] = S_t - K_t P_{t,T} \\
&\Longrightarrow K_t = \frac{S_t}{P_{t,T}} = \mathbb{E}_t^T\left[\frac{S_T}{P_{T,T}}\right] = \mathbb{E}_t^T[S_T]
\end{aligned}
\tag{33}
$$

where $P_{t,T}$ is the price at t of a zero-coupon bond maturing at T. The probability measure associated with this bond numeraire is a variant of risk neutral measure called T-forward measure $\mathbb{Q}^T$. We use $\mathbb{E}_t^T[\cdot]$ to denote the expectation under the measure $\mathbb{Q}^T$.

Futures Contract: At any time from t to T, one may enter or close out the futures contract position at zero cost before it matures. The cashflows maintained by margin account are distributed over the life of the contract rather than coming solely at the end. Suppose at t, one enters into a futures contract at zero cost $V_t = 0$ with a quoted strike $\widetilde{K}_t$ (i.e., the futures price). The contract may incur a cashflow $V_{t+\Delta t} = \widetilde{K}_{t+\Delta t} - \widetilde{K}_t$ after a small time interval Δt due to the market move of $\widetilde{K}_t$. According to the risk neutral pricing formula, we have

$$
0 = D_t V_t = \widetilde{\mathbb{E}}_t[D_{t+\Delta t} V_{t+\Delta t}] = \widetilde{\mathbb{E}}_t\left[D_{t+\Delta t}\left(\widetilde{K}_{t+\Delta t} - \widetilde{K}_t\right)\right] \tag{34}
$$

If we take infinitesimal Δt, then $D_{t+\Delta t} = D_t\, e^{-r_t \Delta t}$ is a known factor at time t, we can move it out of the expectation, and define recursively

$$
\widetilde{K}_t = \widetilde{\mathbb{E}}_t\left[\widetilde{K}_{t+\Delta t}\right] \Longrightarrow \widetilde{K}_t = \widetilde{\mathbb{E}}_t\left[\widetilde{\mathbb{E}}_{t+\Delta t}\left[\widetilde{K}_{t+2\Delta t}\right]\right] = \cdots = \widetilde{\mathbb{E}}_t\left[\cdots \tilde{E}_{T-\Delta t}\left[\widetilde{K}_T\right]\cdots\right] \tag{35}
$$

Based on the *Iterated Conditioning Expectation Theorem* (i.e., If $\mathcal{H}$ holds less information than $\mathcal{G}$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}]$), the strike price of a futures contract can thus be expressed as

$$
\widetilde{K}_t = \widetilde{\mathbb{E}}_t\left[\widetilde{K}_T\right] = \widetilde{\mathbb{E}}_t[S_T] \Longrightarrow \widetilde{K}_t = \widetilde{\mathbb{E}}_t[S_T] \tag{36}
$$

Based on the above derivation, we summarize the forward and futures prices as follows respectively

$$
\begin{aligned}
&K_t = \mathbb{E}_t^T[S_T], \text{ under } T\text{-forward measure } \mathbb{Q}^T \\
&\widetilde{K}_t = \widetilde{\mathbb{E}}_t[S_T], \text{ under risk neutral measure } \mathbb{Q}
\end{aligned}
\tag{37}
$$

The spread between K_t and $\widetilde{K}_t$ can be derived as

$$K_t - \widetilde{K}_t = \frac{S_t}{P_{t,T}} - \widetilde{\mathbb{E}}_t[S_T] = \frac{\widetilde{\mathbb{E}}_t\left[S_T D_{t,T}\right] - \widetilde{\mathbb{E}}_t[S_T]\widetilde{\mathbb{E}}_t\left[D_{t,T}\right]}{P_{t,T}} = \frac{\widetilde{\mathbb{V}}_t\left[S_T, D_{t,T}\right]}{P_{t,T}} \tag{38}$$

where $D_{t,T} = D_T/D_t$ and $\widetilde{\mathbb{V}}_t[\cdot]$ denotes the conditional covariance $\widetilde{\mathbb{V}}[\cdot|\mathcal{F}_t]$ under risk neutral measure. The spread comes from the covariance between the spot price S_t and the discount factor D_t (or indirectly the spot rate r_t). Suppose you long a futures contract on S_T. If the correlation between S_t and r_t is positive (negative between S_t and D_t), the expected return on the excess margin cash is asymmetric and skewed in your favor. This is because when S_t and r_t both go up, the futures contract goes in-the-money, and you can withdraw excess margin cash, which can be deposited at higher rates. However if both S_t and r_t go down, the contract goes out-of-the-money with margin calls but you can fund it at lower rates. On the margin, you invest at higher rates but borrow at lower rates. This asymmetry causes the spread.

2. Spot Rate and Forward Rate

The difference between spot rate and forward rate is that, the spot rate is quoted for an immediate settlement whilst the forward rate is for a settlement in the future. If the interest accrual period is infinitesimal, we call them instantaneous spot rate (i.e., short rate) r_t and instantaneous forward rate $f_{t,T}$, respectively. The r_t is the rate determined at time t for an accrual period between t and $t+dt$, and $f_{t,T}$ is the rate determined at time t for a accrual period in the future between T and $T+dT$. Since the zero coupon bond is a market tradable asset whose payoff at maturity is 1, its price is given by the martingale pricing theorem

$$P_{t,T} = \frac{1}{D_t}\widetilde{\mathbb{E}}_t\left[D_T P_{T,T}\right] = \frac{1}{D_t}\widetilde{\mathbb{E}}_t[D_T] = \widetilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r_u du\right)\right] \tag{39}$$

The instantaneous forward rate can then be defined in terms of the bond price

$$f_{t,T} \equiv -\frac{\partial \log P_{t,T}}{\partial T} = \lim_{\delta\to 0}\frac{P_{t,T} - P_{t,T+\delta}}{P_{t,T+\delta}\delta} \qquad \text{or} \qquad P_{t,T} \equiv \exp\left(-\int_t^T f_{t,u} du\right) \tag{40}$$

Since heuristically $f_{t,T}$ is given by a portfolio of market tradable assets $\left(P_{t,T} - P_{t,T+\Delta T}\right)$ denominated in a numeraire $P_{t,T}$ (divided by a constant ΔT), we should have the following relationship according to (24)

$$f_{t,T} = -\frac{\partial \log P_{t,T}}{\partial T} = \lim_{\delta\to 0}\frac{P_{t,T} - P_{t,T+\delta}}{P_{t,T+\delta}\delta} = \lim_{\delta\to 0}\mathbb{E}_t^{T+\delta}\left[\frac{P_{T,T} - P_{T,T+\delta}}{P_{T,T+\delta}\delta}\right] = \mathbb{E}_t^T\left[f_{T,T}\right] = \mathbb{E}_t^T[r_T] \tag{41}$$

The (40) and (41) are indeed mutually consistent. The proof is as follows

$$\begin{aligned} f_{t,T} &= -\frac{\partial \log P_{t,T}}{\partial T} = -\frac{1}{P_{t,T}}\frac{\partial\widetilde{\mathbb{E}}_t\left[D_{t,T}\right]}{\partial T} = -\frac{1}{P_{t,T}}\widetilde{\mathbb{E}}_t\left[\frac{\partial}{\partial T}\exp\left(-\int_t^T r_u du\right)\right] = \frac{1}{P_{t,T}}\widetilde{\mathbb{E}}_t\left[D_{t,T} r_T\right] \\ &= \frac{1}{P_{t,T}}\mathbb{E}_t^T\left[\frac{P_{t,T}}{P_{T,T}} r_T\right] = \mathbb{E}_t^T[r_T] \end{aligned} \tag{42}$$

where we can move the differential operator into expectation because it is a linear operator and the expectation is a linear function.

In contrast to the instantaneous forward rate with $\Delta T \to 0$, we may consider a simply compounded forward rate $f_{t,T,V}$, which is an interest rate observed at present time t for a future loan period from T to

V for $t < T < V$. By no-arbitrage argument, the rate can be implied from prices of two zero coupon bonds that mature at T and V, respectively

$$f_{t,T,V} = \frac{P_{t,T} - P_{t,V}}{(V-T)P_{t,V}} = \frac{1}{V-T}\left(\frac{P_{t,T}}{P_{t,V}} - 1\right) \tag{43}$$

The interest rate dynamics can be modeled by short rate models, which however emphasize on instantaneous interest rates that are not directly observable in markets. This makes them less straightforward to calibrate to the market traded instruments and more difficult to perform hedging and risk management. To address this issue, one can model the market observable rates (i.e., the simply compounded forward rate, also known as Libor rates) directly using market models, which will be discussed in Chapter 11. Lastly, the arbitrage free condition must be satisfied in these models, which is generalized in the Heath-Jarrow-Morton (HJM) framework that all rates models must comply to. This will be introduced in Chatper 7.

3. OIS Discounting and Multi-Curve Framework

Prior to the 2008 financial crisis, interbank deposits posed little credit/liquidity issues, interbank lending rates (Ibor rates, e.g., Libor, Euribor) were essentially a good proxy for risk free rates. Basis swap spreads were negligible and thereby neglected. A single yield curve constructed out of selected deposit rates, FRA/EDF rates and swap rates served both the cashflow projection and discounting purposes.

During the 2008 financial crisis, the failure of some banks however proved that interbank lending rates (e.g., Libor, Euribor etc.) were not risk-free. Meanwhile there was also significant counterparty credit risk arising from derivative transactions that were not subject to collateral or margin calls. Basis swap spreads greatly widened, and persist to this day. The existence of such significant basis swap spreads reflects the fact that after the crisis the interest rate market has been segmented into sub-areas corresponding to instruments with different underlying rate tenors, characterized by different rate dynamics. Traditional single curve based pricing approach ignores the differences. It mixes different underlying rate tenors and incorporates different rate dynamics, eventually leads to inconsistency.

After the crisis, the market practice has thus evolved to take into account the new market information (e.g., the basis swap spreads, collateralization, etc.), that translate into the additional requirement of homogeneity and funding. The homogeneity requirement means that interest rate derivatives with a given underlying rate tenor must be priced and hedged using vanilla interest rate market instruments with the same underlying. The funding requirement means that the discount rate of any cashflow generated by the derivative must be consistent, by no-arbitrage, with the funding rate associated with that cashflow. Derivatives that trade over-the-counter make use of ISDA agreements to standardize the contract documents. Driven by the crisis, many ISDA agreements have now included a credit support annex (CSA) which is an agreement that outlines permissible credit mitigants for a transaction, such as netting and collateralization in cash. Since standard CSA agreements stipulate daily margination on collateral and the cash collateral earns a return at overnight rate, thereby overnight rate becomes a natural choice for the risk-free discount rate or the funding rate. This is referred to as "OIS discounting" or "CSA discounting" [2].

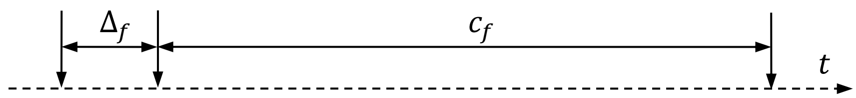
The large spread between risk free rate and interbank lending rate during and after 2008 financial crisis triggered the separation of projection and discount curve in derivative valuation. The traditional single curve used for both cashflow projection and discounting turned out to be obsolete. The markets have since nearly switched to multi-curve framework. In addition to discount curve $P_{t,T}$, there is another curve $\hat{P}_{t,T}$ that serves dedicatedly for cashflow projection (here the accent “hat” denotes a quantity related to projection curve). The quantity $\hat{P}_{t,T}$ acts as a pseudo-discount factor. Interest rate swaps use the curve $\hat{P}_{t,T}$ to estimate floating rates and hence the projected cashflows, which are then discounted by discount curve $P_{t,T}$ to give present value. The spread between the projection curve and the discount curve reflects (at least partially) the liquidity and the credit risk.

3.1. Libor Rates

Libor (London Interbank Offered Rate[1]) is the daily reference rate at which banks borrow large amount of unsecured funds from each other. Libor rates are calculated daily through a survey of a panel of international banks asking how much they would be charged if borrowing cash from other banks (based on estimates rather than actual trade data). The top and bottom quartiles of quotes are excluded, and those left are averaged and made public before noon in London. The rates are produced in 10 currencies for 15 maturities (tenors) ranging from overnight to one year. The Libor is widely used as a reference rate for many financial instruments in both financial markets and commercial fields. Nowadays (as of 2012), At least \$350 trillion notional in derivatives and other financial products are linked to the Libor. There are many other interbank rates, such as Euribor (Euro Interbank Offered Rate), Tibor (Tokyo Interbank Offered Rate), etc., that differs for tenor, fixing mechanics, contribution panel, etc. In general, we will refer to these rates with the generic term “Libor”, discarding further distinctions if not necessary.

[1] https://en.wikipedia.org/wiki/Libor

In USD, Libor rates are quoted as an annualized and simply compounded interest rate. It follows Actual/360 day count convention and modified following with end-of-month business day convention. The schedule of Libor borrowing is sketched in the following diagram (and will be discussed in more detail in next section)



On fixing date f, the borrower and the lender agree on a fixed Libor rate $\hat{L}(f, f_s, f_e)$. The loan takes place in two business days (i.e., spot lag $\Delta_f = 2D$) on value date f_s and repays on maturity date f_e. Two exceptions apply. First, for overnight (O/N) Libor rate, the fixing and value date are the same. Second, if two London business days after fixing date falls on a US holiday, the value date will be rolled forward to the next available business day. The loan accrues interest for a period of the rate tenor. Upon maturity, the borrower repays the principal N plus interest $Nc_f\hat{L}(f, f_s, f_e)$ with $c_f = \frac{\text{\# calendar days between } f_s \text{ and } f_e}{360}$.

A fundamental assumption about the Libor rate $\hat{L}(f, f_s, f_e)$ is that the value of the floating coupon $Nc_f\hat{L}(f, f_s, f_e)$ is a market tradable asset. Its price at t for $t < f$ is computed as a discounted forward coupon, $Nc_f\hat{L}(t, f_s, f_e)P(t, f_e)$, where the forward Libor rate $\hat{L}(t, f_s, f_e)$ is estimated from a projection curve $\hat{P}(t, T)$ as

$$\hat{L}(t, f_s, f_e) = \frac{\hat{P}(t, f_s) - \hat{P}(t, f_e)}{c_f\hat{P}(t, f_e)} \quad \forall \; t < f \tag{44}$$

Since on fixing date f the forward Libor rate converges to its underlying spot, for simplicity we have just denoted the spot Libor rate by $\hat{L}(f, f_s, f_e)$. An implication of the assumption is that the forward Libor rate is a martingale under f_e-forward measure with numeraire $P(t, f_e)$

$$\frac{Nc_f\hat{L}(t, f_s, f_e)P(t, f_e)}{P(t, f_e)} = \mathbb{E}_t^{f_e}\left[\frac{Nc_f\hat{L}(f, f_s, f_e)}{P(f_e, f_e)}\right] \Longrightarrow \hat{L}(t, f_s, f_e) = \mathbb{E}_t^{f_e}\left[\hat{L}(f, f_s, f_e)\right] \tag{45}$$

$$\Rightarrow \frac{\hat{P}(t,f_s)-\hat{P}(t,f_e)}{c_f\hat{P}(t,f_e)} = \mathbb{E}_t^{f_e}\left[\frac{\hat{P}(f,f_s)-\hat{P}(f,f_e)}{c_f\hat{P}(f,f_e)}\right] \Rightarrow \frac{\hat{P}(t,f_s)}{\hat{P}(t,f_e)} = \mathbb{E}_t^{f_e}\left[\frac{\hat{P}(f,f_s)}{\hat{P}(f,f_e)}\right]$$

On the contrary, the *pseudo* zero coupon bond $\hat{P}(t,T)$ cannot be a market tradable asset nor its numeraire-rebased value a martingale. Only the risk free zero coupon bond $P(t,T)$ can serve as a market tradable asset and hence a numeraire.

Since the rate $\hat{L}(f,f_s,f_e)$ has been fixed at f, the coupon payment of $Nc_f\hat{L}(f,f_s,f_e)$ at f_e is equivalent to a payment of $Nc_f\hat{L}(f,f_s,f_e)P(f,f_e)$ at f. Using (24) again, we can derive another martingale under f_s-forward measure

$$\begin{aligned}
&\frac{Nc_f\hat{L}(t,f_s,f_e)P(t,f_e)}{P(t,f_s)} = \mathbb{E}_t^{f_s}\left[\frac{Nc_f\hat{L}(f,f_s,f_e)P(f,f_e)}{P(f,f_s)}\right] \\
&\Rightarrow \frac{P(t,f_e)}{P(t,f_s)}\left(\frac{\hat{P}(t,f_s)}{\hat{P}(t,f_e)}-1\right) = \mathbb{E}_t^{f_s}\left[\frac{P(f,f_e)}{P(f,f_s)}\left(\frac{\hat{P}(f,f_s)}{\hat{P}(f,f_e)}-1\right)\right] \\
&\Rightarrow \frac{P(t,f_e)}{P(t,f_s)}\frac{\hat{P}(t,f_s)}{\hat{P}(t,f_e)} = \mathbb{E}_t^{f_s}\left[\frac{P(f,f_e)}{P(f,f_s)}\frac{\hat{P}(f,f_s)}{\hat{P}(f,f_e)}\right] \\
&\Rightarrow \eta(t,f_s,f_e) = \mathbb{E}_t^{f_s}[\eta(f,f_s,f_e)]
\end{aligned} \tag{46}$$

where we define a multiplicative spread between the projection and the discount curve as

$$\eta(t,f_s,f_e) = \frac{P(t,f_e)}{P(t,f_s)}\frac{\hat{P}(t,f_s)}{\hat{P}(t,f_e)} = \frac{1+c_f\hat{L}(t,f_s,f_e)}{1+c_fL(t,f_s,f_e)} \tag{47}$$

In (47), the $L(t,f_s,f_e)$ is a (pseudo) Libor rate estimated on discount curve similar to (44)

$$L(t,f_s,f_e) = \frac{P(t,f_s)-P(t,f_e)}{c_fP(t,f_e)} \quad \forall \ t < f \tag{48}$$

Since both numerator and denominator are market tradable assets, the $L(t,f_s,f_e)$ is a martingale under f_e-forward measure.

3.2. Interest Rate Swap: Schedule Generation

Payment and fixing schedules play important roles in interest rate modeling and pricing. A fixed for floating interest rate swap exchanges a stream of periodic fixed interest payments with a stream of periodic floating interest payments over a term to maturity. Floating leg of the swap (e.g., fixed in advance and paid in arrears) can be regarded as a portfolio of coupon cashflows paid at a series of scheduled dates. Due to variations in holidays and conventions agreed by parties, the way to compute those dates must be detailed. Here we will use a swap floating leg as an example to explain the schedule generation. Table 3.1 lists some common specifications of a floating leg, which can also apply to a fixed leg with minor modifications.

Table 3.1 Specifications of a typical IRS leg

attribute	symbol	remark/example
trade date[1]	t_0	today
spot date[2]	t_s	$t_s = t_0 \oplus \Delta_s$
spot lag	Δ_s	2D
payment lag[3]	Δ_p	0D
effective date[4]	t_e	$t_e = a_{1,s}$
maturity date[5]	t_m	$t_m = a_{n,e}$
roll day[6]	d	29
payment/reset frequency[7]	P	1M, 6M, 12M
rate index[8]	L	6M Euribor
day count convention	τ	Actual/360
business day convention[9]	B	Modified Following and End of Month
calendar		TARGET, US, UK

[1] Trade date is the day on which the swap is traded, e.g. today.
[2] Spot date is the effective date of a spot starting swap. It is usually calculated as "spot lag" of business days after the trade date. The "$\oplus$" denotes the date increment in business days.
[3] Most swaps have payment lag of "0D". One exception is the overnight index swap (OIS). Depending on conventions, some OIS swaps may not be able to fix its floating rate until the end of each coupon period. This incurs a payment lag.
[4] Effective date is also referred to the start date, the value date or the settlement date. It must be a business day and coincides with the start date of the first coupon period. If $t_e = t_s$, it is usually called spot starting swap (e.g. T+2), while $t_e > t_s$ we have a forward starting swap. In the case where $t_e = t_0$, it is called same day starting swap (e.g. T+0).
[5] Maturity date coincides with the end date of the last coupon period.
[6] Roll day (e.g. an integer between 1 and 31) defines on which day in a month the interest accrual periods start/end. It means that the (unadjusted) dates will be on the given day. In a front-stubbed swap, maturity date must be the business convention adjusted roll day of that month.
[7] Payment/reset frequency define the size of coupon periods for a fixed/floating leg, e.g. Monthly, Quarterly, Semi-Annually, or Annually.
[8] Rate index is the benchmark interest rate that the floating leg payment linked to. The tenor of the rate index is usually the same as the payment/reset frequency of the swap leg.
[9] Business day convention (e.g. Modified Following with adjustment to period end dates) on a calendar (e.g. TARGET, US, UK) adjusts rolled (unadjusted) dates to business days.

Swap specifications vary across currencies and regions, and usually follow the interest rate swap (IRS) market conventions as summarized in Table 3.2 [3].

Table 3.2 Most common vanilla swap conventions

		floating leg			fixed leg	
currency	spot lag	reference	period	convention	period	convention
USD (NY)	2	Libor	3M	ACT/360	6M	30/360
USD (London)	2	Libor	3M	ACT/360	1Y	ACT/360
EUR: 1Y	2	Euribor	3M	ACT/360	1Y	30/360
EUR: > 1Y	2	Euribor	6M	ACT/360	1Y	30/360
GBP: 1Y	0	Libor	3M	ACT/365	1Y	ACT/365
GBP: > 1Y	0	Libor	6M	ACT/365	6M	ACT/365
JPY	2	Tibor	3M	ACT/365	6M	ACT/365
JPY	2	Libor	6M	ACT/360	6M	ACT/365
CHF: 1Y	2	Libor	3M	ACT/360	1Y	30/360
CHF: > 1Y	2	Libor	6M	ACT/360	1Y	30/360

A coupon period of a swap floating leg usually has separate date definitions for rate fixing and for interest accrual. Figure 3.1 depicts a coupon period with typical definitions of dates. Detailed descriptions of the dates are listed in Table 3.3. A coupon period of fixed leg possesses similar ingredients except that the dates defined for rate fixing may be omitted.

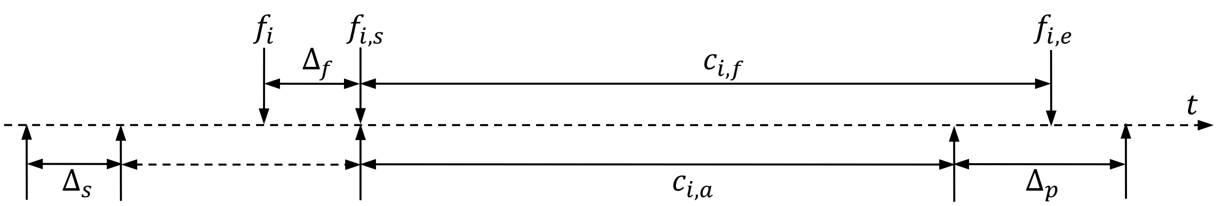


Figure 3.1 One coupon period of a typical swap floating leg (e.g., the i-th period)

Table 3.3 Attributes of an IRS coupon period (e.g., the i-th of the total n periods)

attribute	symbol	description	remark/example
accrual start date[1]	$a_{i,s}$	on which the accrual starts	$a_{1,s} = t_e,\ \ a_{i,s} = a_{i-1,e}$
accrual end date[2]	$a_{i,e}$	on which the accrual ends	$a_{i,e} = B\big(d.(\mathrm{M.Y} - (n-i)P)\big)$
payment date	p_i	on which the payment is made	$p_i = a_{i,e} \oplus \Delta_p$

[1] Formula $a_{i,s} = a_{i-1,e}\ \forall\, i > 1$ means the start date of current accrual period coincides (conventionally) with the end date of previous accrual period.
[2] Formula $d.(\mathrm{M.Y} - (n-i)P)$ means we keep the roll day unchanged and only roll the month and year.

index spot lag[1]	Δ_f	spot lag of benchmark rate index	2D
fixing date	f_i	on which the index rate is fixed	$f_i \oplus \Delta_f = a_{i,s}$
fixing start date[2]	$f_{i,s}$	on which the fixing period starts	$f_{i,s} = a_{i,s}$
fixing end date[3]	$f_{i,e}$	on which the fixing period ends	$f_{i,e} = B(f_{i,s} + L)$
accrual coverage[4]	$c_{i,a}$	accrual period year fraction	$c_{i,a} = \tau(a_{i,s}, a_{i,e})$
fixing coverage[5]	$c_{i,f}$	fixing period year fraction	$c_{i,f} = \tau(f_{i,s}, f_{i,e})$

In general, the schedule is worked out backwards for a front-stubbed swap. Assuming we have a roll day d, the maturity date t_m given as $\mathrm{D.M.Y}$ (e.g., day.month.year) must be consistent with the roll day d such that the business day adjusted roll day in the month of maturity coincides with the maturity date, e.g., $B(d.\mathrm{M.Y}) = \mathrm{D.M.Y}$. The maturity date also marks the end date of the last interest accrual period. For the rest of the periods, their end dates can be deduced using the formula $a_{i,e} = B\big(d.(\mathrm{M.Y} - (n-i)P)\big)$ for $i = n$ to 1, where before adjustment the roll day remains unchanged and the month and year are rolled towards effective date by a period of payment/reset frequency P. In other words, all the dates are first rolled (may result in an invalid date, like February 30) without adjustment and then all the dates are adjusted. There will be a stub period in front if for the first period $t_e \neq B\big(d.(\mathrm{M.Y} - nP)\big)$. The reason the stub period is the first one is that once that period is finished, the swap has the same schedule as a standard one. If the stub was the last period, the swap would never become a standard one. Providing that all the $a_{i,e}$ are available, the rest are trivial and can be deduced using the formulas provided in Table 3.3. Note that the end date of fixing period $f_{i,e}$ corresponding to the benchmark rate index can be slightly different from the end date of floating coupon period $a_{i,e}$. The difference is created by the adjustments due to non-good business days.

3.3. Interest Rate Swap: Valuation

[1] Index spot lag is the spot lag associated with the benchmark rate index. It is usually the same as the swap spot lag, e.g. both are two business days.
[2] Fixing start date usually coincides with the accrual start date within a coupon period.
[3] The L here denotes the tenor of the rate index. The business day adjustment $B(\cdot)$ may vary for the accrual end date and for the fixing end date.
[4] The $\tau(a_{i,s}, a_{i,e})$ is the year fraction between $a_{i,s}$ and $a_{i,e}$ given by day count convention $\tau(\cdot)$.
[5] The day count convention $\tau(\cdot)$ may vary for accrual coverage and for fixing coverage.

In the most common fixed for floating (spot starting) interest rate swap, all the coupon payments are calculated based on the same notional and all the coupons on the fixed leg have the same rate. Following the notation, the present value of the floating leg of a vanilla swap can be calculated as

$$V_{\text{float}}(t_0) = \sum_i c_{i,a} P(t_0, p_i) \mathbb{E}_{t_0}^{p_i}\left[\hat{L}\left(f_i, f_{i,s}, f_{i,e}\right)\right] \approx \sum_i c_{i,a} P(t_0, p_i) \hat{L}\left(t_0, f_{i,s}, f_{i,e}\right) \tag{49}$$

where p_i and $c_{i,a}$ are the payment date and coverage of the i-th accrual period of the floating leg, and the forward Libor rate is estimated from the pseudo discount factor of the projection curve

$$\hat{L}\left(t, f_{i,s}, f_{i,e}\right) = \frac{\hat{P}\left(t, f_{i,s}\right) - \hat{P}\left(t, f_{i,e}\right)}{c_{i,f} \hat{P}\left(t, f_{i,e}\right)} \quad \text{for} \quad t < f_i \tag{50}$$

If we plug in the par swap rate $S(t_0)$ fixed at trade date t_0, the fixed leg of the vanilla swap would have the same present value of the floating leg, which can be calculated as

$$V_{\text{fixed}}(t_0) = S(t_0) \sum_j c_{j,a} P\left(t_0, p_j\right) \tag{51}$$

where p_j and $c_{j,a}$ are the payment date and coverage of the j-th accrual period of the fixed leg. Note that the quantities associated with the floating leg and the fixed leg are differentiated by indices i and j respectively.

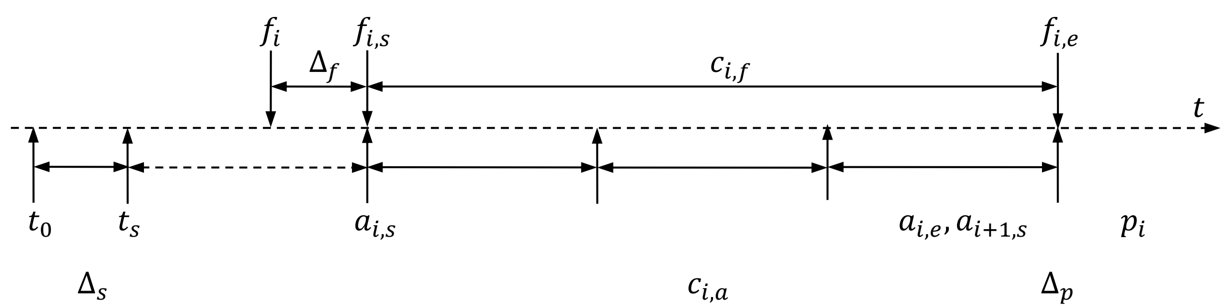


Figure 3.2 The i-th coupon period of a swap with a composition Libor index floating leg

Some swaps are also traded on a compounded basis that aligns the payment of floating leg and fixed leg to reduce the credit risk. For example, as depicted in Figure 3.2, a trade that swaps 1M Libor versus 3M fixed coupon can be quoted with the 1M Libor compounded over three 1M periods and paid quarterly in line with the 3M period. The quantity $V_{1\text{M}}(t_0)$ below is the t_0-value of the 1M Libor interest accrued in the first 1M period and paid at the end of the 3M period. $a_{i,s}=a_{j,s}$ and $a_{i+2,e}=a_{j,e}$. The p_j is the payment date of the j-th fixed coupon and usually we have $p_j=a_{j,e}=a_{i+2,e}$. The PV of the swap can be approximated by

$$
\begin{aligned}
&\frac{V(t_0)}{Nc_{i,a}P(t_0,p_j)}=\mathbb{E}_{t_0}^{p_j}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\prod_{k=i+1}^{i+2}\left(1+c_{k,a}\hat{L}(f_k,f_{k,s},f_{k,e})\right)\right]\\
&\approx \mathbb{E}_{t_0}^{p_j}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\prod_{k=i+1}^{i+2}\frac{\hat{P}(f_k,a_{k,s})}{\hat{P}(f_k,a_{k,e})}\right],\qquad \text{assume } f_{k,s}=a_{k,s},\quad f_{k,e}=a_{k,e}\\
&\approx \mathbb{E}_{t_0}^{a_{i+2,e}}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\prod_{k=i+1}^{i+2}\frac{\hat{P}(f_k,a_{k,s})}{\hat{P}(f_k,a_{k,e})}\right],\qquad \text{assume } P(a_{i+2,e},p_j) \text{ is independent}\\
&= \mathbb{E}_{t_0}^{a_{i+2,e}}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\frac{\hat{P}(f_{i+1},a_{i+1,s})}{\hat{P}(f_{i+1},a_{i+2,e})}\right],\qquad \text{by } \frac{\hat{P}(f_{i+1},a_{i+2,s})}{\hat{P}(f_{i+1},a_{i+2,e})}=\mathbb{E}_{f_{i+1}}^{a_{i+2,e}}\left[\frac{\hat{P}(f_{i+2},a_{i+2,s})}{\hat{P}(f_{i+2},a_{i+2,e})}\right]\\
&= \frac{P(t_0,a_{i+1,s})}{P(t_0,a_{i+2,e})}\mathbb{E}_{t_0}^{a_{i+1,s}}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\frac{\hat{P}(f_{i+1},a_{i+1,s})}{\hat{P}(f_{i+1},a_{i+2,e})}P(a_{i+1,s},a_{i+2,e})\right]\\
&= \frac{P(t_0,a_{i+1,s})}{P(t_0,a_{i+2,e})}\mathbb{E}_{t_0}^{a_{i+1,s}}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\eta(f_{i+1},a_{i+1,s},a_{i+2,e})\right]\\
&\approx \frac{P(t_0,a_{i+1,s})}{P(t_0,a_{i+2,e})}\mathbb{E}_{t_0}^{a_{i+1,s}}\left[\hat{L}(f_i,f_{i,s},f_{i,e})\right]\mathbb{E}_{t_0}^{a_{i+1,s}}\left[\eta(f_{i+1},a_{i+1,s},a_{i+2,e})\right],\qquad \text{assume } \hat{L}_i\perp\eta\\
&\approx \frac{P(t_0,a_{i+1,s})}{P(t_0,a_{i+2,e})}\hat{L}(t_0,f_{i,s},f_{i,e})\eta(t_0,a_{i+1,s},a_{i+2,e}),\qquad \text{by } a_{i+1,s}=f_{i,e}\\
&= \frac{P(t_0,a_{i+1,s})}{P(t_0,a_{i+2,e})}\hat{L}(t_0,f_{i,s},f_{i,e})\frac{P(t_0,a_{i+2,e})}{P(t_0,a_{i+1,s})}\frac{\hat{P}(t_0,a_{i+1,s})}{\hat{P}(t_0,a_{i+2,e})}\\
&= \hat{L}(t_0,f_{i,s},f_{i,e})\frac{\hat{P}(t_0,a_{i+1,s})}{\hat{P}(t_0,a_{i+2,e})}\\
&\Longrightarrow V(t_0)\approx Nc_{i,a}\hat{L}(t_0,f_{i,s},f_{i,e})\frac{\hat{P}(t_0,a_{i+1,s})}{\hat{P}(t_0,a_{i+2,e})}P(t_0,p_j)
\end{aligned}
\tag{52}
$$

3.4. Forward Rate Agreement

Forward Rate Agreement (FRA) is a forward contract traded over-the-counter (OTC) between two parties to lock in a forward rate today, for money they intend to borrow or lend sometime in the future. It can be simply taken as a one-period forward starting fixed for floating interest rate swap. At trade date, two parties enter into a FRA to agree on a coupon rate, a start period and a reference index. For example, an $n \times m$ FRA (reads as "n-against-m-month" FRA) contract has a start period of nM and an end period of $(n+m)M$, which is the start period plus the index tenor (i.e a $1M$ start period and a $6M$ tenor give a $7M$ end period). The accrual start date (accrual end date) is computed from spot date by adding the start period (end period) and then adjusted by business day convention. The fixing start date coincides with the accrual start date. The fixing end date is computed from the fixing start date by adding the index tenor ($= mM$) and then adjusted by business day convention. The fixing date (or exercise date) is the spot lag before the fixing start date. At accrual start date (i.e., value date, settlement date), the difference between the coupon rate and the index rate is then discounted back from the accrual end date (i.e., maturity date) to value date at the index rate and cash settled on the value date rather than the maturity date to reduce credit risk.

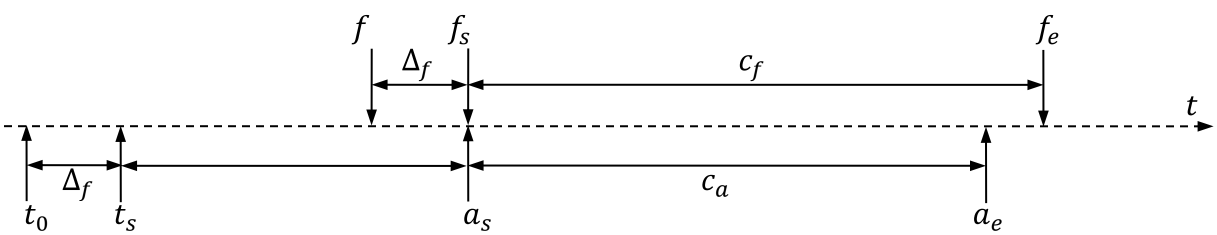


Figure 3.3 Schedule of a FRA

Table 3.4 Attributes of a FRA contract

attribute	symbol	description	remark/example
trade date	t_0	on which the FRA is traded	today
spot date	t_s	trade date plus index spot lag	$t_s = t_0 \oplus \Delta_f$
index spot lag	Δ_f	spot lag of reference index	2D
accrual start date[1]	a_s	on which the accrual starts	$a_s = B(t_s + nM)$
accrual end date[2]	a_e	on which the accrual ends	$a_e = B(t_s + (n+m)M)$

[1] Start period is usually specified in number of months, e.g. nM.
[2] The tenor of rate index is mM.

fixing date	f	on which the index rate is fixed	$f \oplus \Delta_f = a_s$
fixing start date	f_s	on which the fixing period starts	$f_s = a_s$
fixing end date[1]	f_e	on which the fixing period ends	$f_e = B(f_s + L)$
accrual coverage	c_a	accrual period year fraction	$c_a = \tau(a_s, a_e)$
fixing coverage	c_f	fixing period year fraction	$c_f = \tau(f_s, f_e)$

The FRA is settled at f_s with a payoff

$$\frac{Nc_a\omega\left(\hat{L}(f, f_s, f_e) - R_{\text{FRA}}(t)\right)}{1 + c_a\hat{L}(f, f_s, f_e)} \tag{53}$$

where $R_{\text{FRA}}(t)$ is the contractual FRA rate fixed at time t. Since the rate $\hat{L}(f, f_s, f_e)$ has been fixed at f, the payoff settled at f_s is equivalent to the same payoff discounted by $P(f, f_s)$ and settled at f, the time t value of the FRA would be

$$\begin{aligned} V_{\text{FRA}}(t) &= P(t, f_e)\mathbb{E}_t^{f_e}\left[\frac{Nc_a\omega\left(\hat{L}(f, f_s, f_e) - R_{\text{FRA}}(t)\right)}{1 + c_a\hat{L}(f, f_s, f_e)}\frac{P(f, f_s)}{P(f, f_e)}\right] \\ &= N\omega P(t, f_e)\mathbb{E}_t^{f_e}\left[c_a\left(\hat{L}(f, f_s, f_e) - R_{\text{FRA}}(t)\right)\frac{1 + c_f L(f, f_s, f_e)}{1 + c_a\hat{L}(f, f_s, f_e)}\right] \\ &= N\omega P(t, f_e)\left(1 + c_f L(t, f_s, f_e) - \left(1 + c_a R_{\text{FRA}}(t)\right)\mathbb{E}_t^{f_e}\left[\frac{1 + c_f L(f, f_s, f_e)}{1 + c_a\hat{L}(f, f_s, f_e)}\right]\right) \end{aligned} \tag{54}$$

An equilibrium FRA rate would give a vanishing initial value and hence

$$R_{\text{FRA}}(t) = \frac{1}{c_a}\left(\frac{1 + c_f L(t, f_s, f_e)}{\mathbb{E}_t^{f_e}\left[\frac{1 + c_f L(f, f_s, f_e)}{1 + c_a\hat{L}(f, f_s, f_e)}\right]} - 1\right) \tag{55}$$

The expectation of the ratio depends on the joint distribution of the rates. We may assume shifted lognormal martingales for the two rates $L(f, f_s, f_e)$ and $\hat{L}(f, f_s, f_e)$ under f_e-forward measure

$$\frac{d\left(1 + c_f L(t, f_s, f_e)\right)}{1 + c_f L(t, f_s, f_e)} = \sigma' dW_t^{f_e}, \qquad \frac{d\left(1 + c_a\hat{L}(t, f_s, f_e)\right)}{1 + c_a\hat{L}(t, f_s, f_e)} = \hat{\sigma}' dW_t^{f_e}, \qquad dW_t^{f_e} dW_t^{f_e\prime} = \rho dt \tag{56}$$

[1] The L here denotes the tenor of the rate index, e.g. $L = mM$.

which have simple solutions like

$$
\begin{aligned}
1 + c_f L(f, f_s, f_e) &= \left(1 + c_f L(t, f_s, f_e)\right) \exp\left(\sigma'\left(W_f - W_t\right) - \frac{1}{2}\sigma'\rho\sigma(f - t)\right) \\
1 + c_a \hat{L}(f, f_s, f_e) &= \left(1 + c_a \hat{L}(t, f_s, f_e)\right) \exp\left(\hat{\sigma}'\left(W_f - W_t\right) - \frac{1}{2}\hat{\sigma}'\rho\hat{\sigma}(f - t)\right)
\end{aligned} \tag{57}
$$

The expectation can then be estimated

$$
\begin{aligned}
&\mathbb{E}_t^{f_e}\left[\frac{1 + c_f L(f, f_s, f_e)}{1 + c_a \hat{L}(f, f_s, f_e)}\right] = \frac{1 + c_f L(t, f_s, f_e)}{1 + c_a \hat{L}(t, f_s, f_e)} \exp\left(\hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t)\right) \\
&\Longrightarrow R_{\text{FRA}}(t) = \frac{1}{c_a}\left(\left(1 + c_a \hat{L}(t, f_s, f_e)\right) \exp\left(\hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t)\right) - 1\right) \\
&\approx \frac{1}{c_a}\left(\left(1 + c_a \hat{L}(t, f_s, f_e)\right)\left(1 + \hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t)\right) - 1\right) \\
&= \frac{1}{c_a}\left(1 + c_a \hat{L}(t, f_s, f_e) + \left(1 + c_a \hat{L}(t, f_s, f_e)\right)\hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t) - 1\right) \\
&= \frac{1}{c_a}\left(c_a \hat{L}(t, f_s, f_e) + \left(1 + c_a \hat{L}(t, f_s, f_e)\right)\hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t)\right) \\
&\approx \hat{L}(t, f_s, f_e) + \left(1 + c_a \hat{L}(t, f_s, f_e)\right)\hat{\sigma}'\rho(\hat{\sigma} - \sigma)
\end{aligned} \tag{58}
$$

where the extra term $\exp\left(\hat{\sigma}'\rho(\hat{\sigma} - \sigma)(f - t)\right)$ is the convexity adjustment. For typical post credit crunch market situations, the actual size of the convexity adjustment results to be below 1 bp, even for very long maturities [2].

3.5. Short Term Interest Rate Futures

Specifically, we discuss the Libor based short term interest rate (STIR) futures that are traded at exchanges and subject to margining process. The instruments share the same settlement mechanism but differ in notional, underlying Libor index and exchange where they are quoted. One typical example of the STIR futures is the Eurodollar futures (EDF), which is based on 3M USD Libor rate, reflecting the rate for a 3-month $1,000,000 notional offshore deposit. EDF is basically the futures equivalent of FRA that allows holders to lock in a forward 3M Libor at an earlier time. Because EDFs are exchange-traded standardized instrument, they offer greater liquidity and lower transaction costs, despite cannot be customized like FRA's. Furthermore, EDFs are marginated, there is virtually no credit risk, as any gains or losses are daily settled.

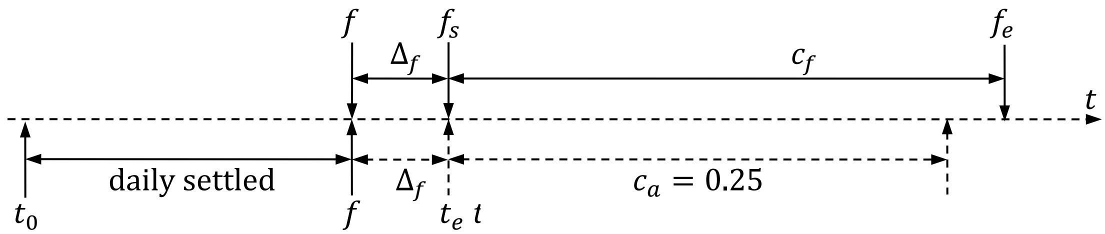


Figure 3.4 Schedule of Eurodollar Futures

Table 3.5 Attributes of a Eurodollar Futures contract

attribute	symbol	description	remark/example
nominal amount	N	used for margin calculation	\$1,000,000
trade date	t_0	on which the EDF is traded	today
rate index	L	underlying rate index	3M USD Libor
index spot lag	Δ_f	spot lag of reference index	2D
settlement date	t_e	start date of underlying Libor	$t_e = f \oplus \Delta_f$
index fixing date[1]	f	on which the index rate is fixed	$f \oplus \Delta_f = f_s$
fixing start date	f_s	on which the fixing period starts	$f_s = t_e$
fixing end date	f_e	on which the fixing period ends	$f_e = B(f_s + L)$
accrual factor	c_a	90 days in Actual/360 convention	$c_a = 0.25$
fixing coverage	c_f	fixing period year fraction	$c_f = \tau(f_s, f_e)$

There are 40 quarterly EDF contracts, spanning 10 years, plus 4 more for nearest serial (non-quarterly) months that are listed at all times. The quarterly EDF contracts have delivery months of March, June, September and December. Of each contract, the final settlement day is the third Wednesday of the settlement month. The last trading day is two business days prior to the final settlement date. The EDFs are quoted in terms of the "IMM index". From the point of view of the counterparty paying the floating rate, the price at time t reads $100\big(1 - R_{\text{EDF}}(t)\big)$ for a traded futures rate $R_{\text{EDF}}(t)$ (e.g., quoted at 99.25 for $R_{\text{EDF}}(t) = 0.75\%$). Upon fixing date f, the futures rate must converge to the official fixing of 3M

[1] Index fixing date is also the last trade date. The EDF contract is daily settled between the trade date and the fixing date.

Libor rate such that $R_{\text{EDF}}(f) = \hat{L}(f, f_s, f_e)$, and hence the final settlement price becomes $100\left(1 - \hat{L}(f, f_s, f_e)\right)$.

In order to provide a rule to compute the margin for the futures contracts, the value of the EDF contracts is defined as

$$V_{\text{EDF}}(t) = Nc_a\big(1 - R_{\text{EDF}}(t)\big) = \$250{,}000 \times \big(1 - R_{\text{EDF}}(t)\big) \tag{59}$$

Namely, one basis point (0.01%) fluctuation in futures rate would result in \$25 movement in the contract value. For a given closing price $R_{\text{EDF}}(t)$ (as published by the exchange), the daily margin paid for one EDF contract can be calculated as the closing price minus the reference price $R_{\text{EDF}}(t-1)$ multiplied by the nominal amount and then by the accrual factor

$$\Delta_{\text{EDF}}(t-1, t) = V_{\text{EDF}}(t-1) - V_{\text{EDF}}(t) = Nc_a\big(R_{\text{EDF}}(t) - R_{\text{EDF}}(t-1)\big) \tag{60}$$

where the reference price is the trade price on the trade date and the previous closing price on the subsequent dates. Because the daily settlement mechanism of EDF, market quote of EDF is slightly higher than that of FRA. To infer forward rates from EDF rates, it is necessary to make convexity adjustment, which can be quantified by forward-futures spread as discussed in section 0. The exact value assigned to the convexity adjustment however depends on a model of future evolution of interest rates. This will be discussed in more details in the following chapters.

3.6. Overnight Index Swap

The overnight indexed swaps (OIS) exchange a leg of fixed payments for a leg of floating payments linked to an overnight index. Table 3.6 list the overnight indices of the four major currencies. The start date of the swap is the trade date plus a spot lag (e.g., most commonly two business days). The payments on the fixed leg are regularly spaced. Most of the OIS have one payment if shorter than one year and a 1Y period for longer swaps. The payments on the floating leg are also regularly spaced, usually on the same dates as the fixed leg. The amount paid on the floating leg is determined by daily compounding the overnight rates. The payment is usually not done on the end of period date, but at a certain lag after the last fixing publication date. The reason of the lag is that the actual amount is only known at the very end of the period; the payment lag allows for a smooth settlement. Table 3.7 shows typical OIS conventions for major currencies. Figure 3.5 and Table 3.8 depict one coupon period of OIS floating leg. It is assumed that there are in total K overnight rate fixings in the i-th coupon period. Clearly, we have $a_{i,s} = d_{1,s}$ and $a_{i,e} = d_{K,e}$.

Table 3.6 Overnight indices of the four major currencies

Currency	Name	Day Count Convention	Publication Time
USD	Effective Fed Funds	ACT/360	morning of end date
EUR	EONIA	ACT/360	evening of start date
JPY	TONAR	ACT/365	morning of end date
GBP	SONIA	ACT/365	evening of start date

Table 3.7 Overnight index swap conventions

		Fixed Leg		Floating Leg			
Currency	Spot Lag[1]	Frequency	Convention	Reference	Frequency	Convention	Pay Lag[2]
USD ≤ 1Y	2	tenor	ACT/360	Fed Fund	tenor	ACT/360	2
USD > 1Y	2	1Y	ACT/360	Fed Fund	1Y	ACT/360	2
EUR ≤ 1Y	2	tenor	ACT/360	EONIA	tenor	ACT/360	2
EUR > 1Y	2	1Y	ACT/360	EONIA	1Y	ACT/360	2
JPY ≤ 1Y	2	tenor	ACT/365	TONAR	tenor	ACT/365	1
JPY > 1Y	2	1Y	ACT/365	TONAR	1Y	ACT/365	1
GBP ≤ 1Y	0	tenor	ACT/365	SONIA	tenor	ACT/365	1
GBP > 1Y	0	1Y	ACT/365	SONIA	1Y	ACT/365	1

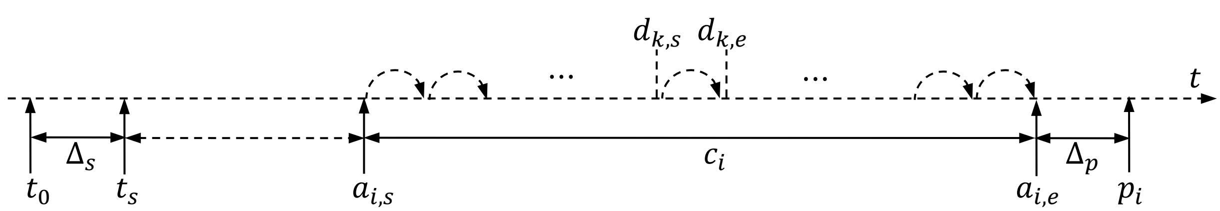


Figure 3.5 One coupon period of overnight index swap floating leg

Table 3.8 Attributes of an OIS floating coupon period (e.g., the i-th of the total n periods)

attribute	symbol	description	remark/example
accrual start date	$a_{i,s}$	start date of i-th period	similar to vanilla IRS

[1] The spot lag is the lag in days between the trade date and the swap start date.
[2] The pay lag is the lag in days between the last fixing publication and the payment.

accrual end date	$a_{i,e}$	end date of i-th period	similar to vanilla IRS
accrual coverage[1]	c_i	coverage of i-th period	$c_i = \tau(a_{i,s}, a_{i,e})$
payment date	p_i	payment date of i-th period	$p_i = a_{i,e} \oplus \Delta_p$
number of fixings	K	total number of fixings in i-th period	$1 \le k \le K$
fixing date	d_k	publication date of k-th O/N rate	$d_k = d_{k,s}$ or $d_{k,e}$
fixing start date	$d_{k,s}$	start date of k-th O/N rate	$d_{k,s} = d_{k-1,e}$
fixing end date	$d_{k,e}$	end date of k-th O/N rate	$d_{k,e} = d_{k,s} \oplus 1D$

Following the notation in Figure 3.5, the present value of the i-th period of the OIS floating leg can be calculated as

$$\begin{aligned} V_{\text{float},i}(t_0) &= P(t_0,p_i)\mathbb{E}_{t_0}^{p_i}\left[\prod_k \Big(1 + r\big(d_{k,f}, d_{k,s}, d_{k,e}\big)\tau\big(d_{k,s}, d_{k,e}\big)\Big) - 1\right] \\ &= P\big(t_0, a_{i,e}\big)\mathbb{E}_{t_0}^{a_{i,e}}\left[P\big(a_{i,e}, p_i\big)\prod_k \Big(1 + r\big(d_{k,f}, d_{k,s}, d_{k,e}\big)\tau\big(d_{k,s}, d_{k,e}\big)\Big)\right] - P(t_0,p_i) \\ &= P\big(t_0, a_{i,e}\big)\mathbb{E}_{t_0}^{a_{i,e}}\left[P\big(a_{i,e}, p_i\big)\prod_{k=1}^{K} \frac{P\big(d_{k,f}, d_{k,s}\big)}{P\big(d_{k,f}, d_{k,e}\big)}\right] - P(t_0,p_i) \end{aligned} \tag{61}$$

Where $r\big(d_{k,f}, d_{k,s}, d_{k,e}\big)$ is the overnight rate fixed at $d_{k,f}$ for one business day period from $d_{k,s}$ to $d_{k,e}$. Its value must be consistent with the discounting curve and can be estimated from the curve by

$$r\big(d_{k,f}, d_{k,s}, d_{k,e}\big) = \frac{P\big(d_{k,f}, d_{k,s}\big) - P\big(d_{k,f}, d_{k,e}\big)}{P\big(d_{k,f}, d_{k,e}\big)\tau\big(d_{k,s}, d_{k,e}\big)} \tag{62}$$

By assuming independence (i.e., ignoring the small convexity) between $P\big(a_{i,e}, p_i\big)$ and $\prod_{k=1}^{K}\frac{P(d_{k,f},d_{k,s})}{P(d_{k,f},d_{k,e})}$, the (61) simplifies to

$$\begin{aligned} V_{\text{float},i}(t_0) &\approx P\big(t_0, a_{i,e}\big)\mathbb{E}_{t_0}^{a_{i,e}}\big[P\big(a_{i,e}, p_i\big)\big]\mathbb{E}_{t_0}^{a_{i,e}}\left[\prod_{k=1}^{K} \frac{P\big(d_{k,f}, d_{k,s}\big)}{P\big(d_{k,f}, d_{k,e}\big)}\right] - P(t_0,p_i) \\ &= P(t_0,p_i)\mathbb{E}_{t_0}^{a_{i,e}}\left[\prod_{k=1}^{K} \frac{P\big(d_{k,f}, d_{k,s}\big)}{P\big(d_{k,f}, d_{k,e}\big)}\right] - P(t_0,p_i) \end{aligned} \tag{63}$$

[1] Most OIS have one payment if shorter than 1Y and a 1Y period for longer swaps. Payments on floating leg are also regularly spaced, usually on the same dates as the fixed leg.

where the expectation can be estimated as follows by repeatedly using the *tower rule*

$$
\begin{aligned}
&\mathbb{E}_{t_0}^{a_{i,e}}\left[\prod_{k=1}^{K}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]=\mathbb{E}_{t_0}^{a_{i,e}}\left[\mathbb{E}_{d_{K-1,f}}^{a_{i,e}}\left[\frac{P\left(d_{K,f},d_{K,s}\right)}{P\left(d_{K,f},d_{K,e}\right)}\right]\prod_{k=1}^{K-1}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]\\
&=\mathbb{E}_{t_0}^{a_{i,e}}\left[\frac{P\left(d_{K-1,f},d_{K,s}\right)}{P\left(d_{K-1,f},a_{i,e}\right)}\prod_{k=1}^{K-1}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]\\
&=\mathbb{E}_{t_0}^{a_{i,e}}\left[\mathbb{E}_{d_{K-2,f}}^{a_{i,e}}\left[\frac{P\left(d_{K-1,f},d_{K,s}\right)}{P\left(d_{K-1,f},a_{i,e}\right)}\frac{P\left(d_{K-1,f},d_{K-1,s}\right)}{P\left(d_{K-1,f},d_{K-1,e}\right)}\right]\prod_{k=1}^{K-2}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]\\
&=\mathbb{E}_{t_0}^{a_{i,e}}\left[\frac{P\left(d_{K-2,f},d_{K-1,s}\right)}{P\left(d_{K-2,f},a_{i,e}\right)}\prod_{k=1}^{K-2}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]=\cdots\\
&=\mathbb{E}_{t_0}^{a_{i,e}}\left[\frac{P\left(d_{K-l,f},d_{K-l+1,s}\right)}{P\left(d_{K-l,f},a_{i,e}\right)}\prod_{k=1}^{K-l}\frac{P\left(d_{k,f},d_{k,s}\right)}{P\left(d_{k,f},d_{k,e}\right)}\right]=\cdots=\frac{P\left(t_0,a_{i,s}\right)}{P\left(t_0,a_{i,e}\right)}
\end{aligned}
\tag{64}
$$

Finally, the present value of the i-th period of the OIS floating leg is

$$
V_{\text{float},i}(t_0)\approx P(t_0,p_i)\left(\frac{P\left(t_0,a_{i,s}\right)}{P\left(t_0,a_{i,e}\right)}-1\right) \tag{65}
$$

and the present value of the j-th period of the OIS fixed leg is

$$
V_{\text{fixed},j}(t_0)=R(t_0)\sum_j\tau\left(a_{j,s},a_{j,e}\right)P\left(t_0,p_j\right) \tag{66}
$$

where $R(t_0)$ is the OIS swap rate observed at t_0. The par swap rate for a single payment OIS with $i=j=1$ would be

$$
V_{\text{float},i}(t_0)=V_{\text{fixed},j}(t_0)\Longrightarrow R(t_0)=\frac{P\left(t_0,a_{i,s}\right)-P\left(t_0,a_{i,e}\right)}{P\left(t_0,a_{i,e}\right)\tau\left(a_{j,s},a_{j,e}\right)} \tag{67}
$$

If multiple payments are involved, the par swap rate would be

$$
V_{\text{float}}(t_0)\approx\sum_i P(t_0,p_i)\left(\frac{P\left(t_0,a_{i,s}\right)}{P\left(t_0,a_{i,e}\right)}-1\right),\qquad V_{\text{fixed}}(t_0)=R(t_0)\sum_j\tau\left(a_{j,s},a_{j,e}\right)P\left(t_0,p_j\right) \tag{68}
$$

$$\Longrightarrow R(t_0) = \frac{\sum_i P(t_0, p_i)\left(\frac{P(t_0, a_{i,s})}{P(t_0, a_{i,e})} - 1\right)}{\sum_j \tau(a_{j,s}, a_{j,e}) P(t_0, p_j)}$$

As mentioned before, in overnight index swaps, the coupon periods and the day count conventions in general coincide for both floating and fixed legs. However here we still use sub-script i and j to differentiate the quantities of the two legs.

3.7. Interest Rate Tenor Basis Swap

The floating-for-floating tenor basis swaps (IRBS) exchange two floating legs in the same currency, tied to two Libor indices of different tenors. The quoting convention is to quote the spread on the shorter tenor leg (e.g., denoted by index i), in such a way that the spread is positive. Following the notation in section 3.2, the present value of the two legs are

$$\begin{aligned} V_{\text{short\_tenor}}(t_0) &= \sum_i \left(\hat{L}(t_0, f_{i,s}, f_{i,e}) + \mu\right) c_{i,a} P(t_0, p_i) \\ V_{\text{long\_tenor}}(t_s) &= \sum_j \hat{L}(t_0, f_{j,s}, f_{j,e}) c_{j,a} P(t_s, p_j) \end{aligned} \tag{69}$$

For example, suppose you trade a swap USD Libor 3M vs USD Libor 6M quoted at 12 (bps) for ten millions paying three months Libor. You will pay on a quarterly basis the USD Libor three months rate plus the spread of 12 bps multiplied by the relevant accrual factor and the notional and receive on a semi-annual basis the USD Libor six months rate without any spread.

This is the conventions for almost all currencies, with the notable exception of EUR. In EUR, the basis swap are conventionally quoted as two swaps. A quote of Euribor 3M vs Euribor 6M quoted at 12 (bps) for ten millions paying the three months has the following meaning. You enter with the counterpart into two swaps fixed against Euribor. In the first swap you receive a fixed rate and pay the 3M Euribor. In the second swap, you pay the same fixed rate plus the 12 bps spread and receive the 6M Euribor. Note that with that convention the spread is paid on an annual basis, like the standard fixed leg of a fixed versus

Libor swap. Even if the quote refers to the spread of a 3M versus 6M swap, the actual spread is paid annually with the fixed leg convention.

$$\begin{aligned}
&V_{\text{float}}(t_s) = -\sum_i \hat{L}_{t_s,i} c_{i,a} P(t_s, p_i)\,, \qquad V_{\text{fix}}(t_s) = S \sum_k c_{k,a} P(t_s, p_k) \\
&V_{\text{fix}}(t_s) = -(S+\mu) \sum_k c_{k,a} P(t_s, p_k)\,, \qquad V_{\text{float}}(t_s) = \sum_j \hat{L}_{t_s,j} c_{j,a} P\big(t_s, p_j\big) \\
&V_{\text{float}}(t_s) = \sum_j \hat{L}_{t_s,j} c_{j,a} P\big(t_s, p_j\big) - \sum_i \hat{L}_{t_s,i} c_{i,a} P(t_s, p_i) - \mu \sum_k c_{k,a} P(t_s, p_k)
\end{aligned} \tag{70}$$

The composition of Libor index described in section 3 is not restricted fixed for Libor swaps. Some basis swaps are also traded on a compounded basis to align the payment on both legs. For example a basis swap one month Libor versus three months Libor can be quoted with the one month Libor compounded over three periods and paid quarterly in line with the three months period. Note that the exact convention on the spread compounding needs to be indicated for the trade. The composition of the shorter tenor leg is currently the standard in USD.

3.8. Floating-Floating Cross Currency Swap

The most common cross currency swaps exchange two floating legs that are linked to Libor indices of the same tenor [4]. The notional of the two legs differs as they are in different currencies. The notional on one leg is usually the notional on the other leg translated in the other currency through an exchange rate. The rate is often the exchange rate at the moment of the trade as agreed between the parties. The notional is paid on both legs at the start and at the end of the swap. In each period, one leg pays Libor flat (usually USD) and the other pays Libor plus a fixed spread. The swap is known as constant notional cross currency swap (CNCCS) as the initially agreed notional amounts of both legs stay unchanged throughout the lifetime of the swap. However, in view of the elevated credit exposure due to deviation in exchange rate, markets (especially in G10 currencies) are in favor of cross currency swaps with exchange rate reset, known as mark-to-market cross currency swap (MtMCCS). In such swap, the notional of the leg which pays Libor flat (usually USD) is reset at the start of the Libor calculation period based on the spot exchange rate at that time. The notional and spread of the other leg is kept constant throughout the contract period.

3.8.1. Constant Notional Cross Currency Swap

The CNCCS, e.g., a EUR (denoted by X) for USD (denoted by \$) swap, is generally collateralized in USD. The market quotes the cross currency swap in term of a basis spread μ applied to the EUR leg. The notional N_X and $N_\$$ are determined by the spot exchange rate $S_{X\$}(t_0)$ fixed at t_0. The EUR leg and the USD leg can then be valued at par using the formula below

$$\begin{aligned}
V_X(t_0) &= N_X\left(-P_X(t_0,t_e)+P_X(t_0,t_m)+\sum_j\left(\hat{L}_X\left(t_0,f_{j,s},f_{j,e}\right)+\mu\right)c_{j,a}P_X\left(t_0,p_j\right)\right)\\
V_\$(t_0) &= N_\$\left(-P_\$(t_0,t_e)+P_\$(t_0,t_m)+\sum_i\hat{L}_\$\left(t_0,f_{i,s},f_{i,e}\right)c_{i,a}P_\$(t_0,p_i)\right)\\
N_\$ &= N_X S_{X\$}(t_0)
\end{aligned} \tag{71}$$

where $P_\$(t,T)$ is the USD OIS discounting curve and $P_X(t,T)$ is the CSA discounting curve. Given a term structure of the basis spread μ, we are able to bootstrap the CSA discounting curve $P_X(t,T)$ that discounts cashflows paid in EUR but collateralized in USD.

3.8.2. Mark-to-Market Cross Currency Swap

The MtMCCS differs from CNCCS by resetting the notional on USD leg at start of each coupon period. The nature of the notional reset on USD leg allows us to view the MtMCCS swap as a portfolio of forward start (except for the first period, which is spot start) single period cross currency swaps. For example, at start of the i-th period, the USD leg pays $N_X S_{X\$}(f_i)$ amount at $a_{i,s}$ and receives $N_X S_{X\$}(f_i)\left(1+c_{i,a}\hat{L}_\$\left(f_i,f_{i,s},f_{i,e}\right)\right)$ at $a_{i,e}$. This translates into the following valuation formula for the USD leg

$$V_\$(t_0)=\sum_i P_\$\left(t_0,a_{i,e}\right)\mathbb{E}_{t_0}^{a_{i,e}}\left[N_\$(f_i)\left(1+c_{i,a}\hat{L}_\$\left(f_i,f_{i,s},f_{i,e}\right)\right)\right]-P_\$\left(t_0,a_{i,s}\right)\mathbb{E}_{t_0}^{a_{i,s}}\left[N_\$(f_i)\right] \tag{72}$$

$$
\begin{aligned}
&= N_X \sum_i P_\$(t_0, f_i)\mathbb{E}_{t_0}^{f_i}\left[S_{X\$}(f_i)\left(P_\$\big(f_i, a_{i,e}\big)\left(1 + c_{i,a}\hat{L}_\$\big(f_i, f_{i,s}, f_{i,e}\big)\right) - P_\$\big(f_i, a_{i,s}\big)\right)\right] \\
&\approx N_X \sum_i \mathbb{E}_{t_0}^{f_i}[S_{X\$}(f_i)]P_\$(t_0, f_i)\mathbb{E}_{t_0}^{f_i}\left[P_\$\big(f_i, a_{i,e}\big)\left(1 + c_{i,a}\hat{L}_\$\big(f_i, f_{i,s}, f_{i,e}\big)\right) - P_\$\big(f_i, a_{i,s}\big)\right] \\
&\approx N_X \sum_i F_{X\$}(t_0, f_i)\left(P_\$\big(t_0, a_{i,e}\big)\left(1 + c_{i,a}\hat{L}_\$\big(t_0, f_{i,s}, f_{i,e}\big)\right) - P_\$\big(t_0, a_{i,s}\big)\right)
\end{aligned}
$$

where N_X is the constant notional on EUR leg and the resetting notional on USD leg is given by $N_\$(t) = N_X S_{X\$}(t)$. There are two approximations in (72). The first comes from an assumption that the exchange rate $S_{X\$}(t)$ is independent of rates (of course this will introduce convexity, but normally it is small). The last is negligible and is resulted from ignoring the time difference between $a_{i,e}$ and $f_{i,e}$. The EUR leg has a constant notional and hence retains the same expression as in (71)

$$
V_X(t_0) = N_X\left(-P_X(t_0, t_e) + P_X(t_0, t_m) + \sum_j \left(\hat{L}_X\big(t_0, f_{j,s}, f_{j,e}\big) + \mu\right)c_{j,a}P_X\big(t_0, p_j\big)\right) \tag{73}
$$

4. Interest Rate Cap/Floors and Swaptions

In this section, we introduce two main rates instruments liquidly traded in the markets: Cap/Floors and Swaptions. To be more illustrative, the introduction will rely on a simplified single-curve based definition of fixed-to-floating interest rate swap commonly found in many textbooks. The multi-curve based version varies slightly and will be introduced in due course in the subsequent chapters.

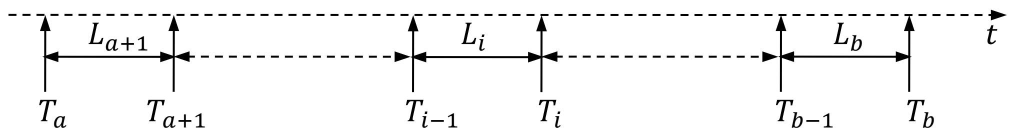


Figure 4.1 Simple schedule definition of a floating leg

Let us consider a sequence of dates, i.e., a payment schedule $T_a < T_{a+1} < \cdots < T_{b-1} < T_b$, such that the time points are approximately equally spaced by a fixed period, e.g., 3M. A forward Libor rate $L_{t,i}\ \ \forall\ i = a+1, \cdots, b$ prevailing at time $t \le T_{i-1}$ is associated with a FRA that starts at T_{i-1} and matures at T_i for a period from T_{i-1} to T_i. By (43), the forward Libor reads

$$L_{t,i} = \frac{P_{t,i-1} - P_{t,i}}{P_{t,i}\tau_i} \tag{74}$$

where τ_i denotes the year fraction between T_{i-1} and T_i given by a day count convention (e.g., Act/360). The forward Libor $L_{t,i}$ becomes a spot rate $L_{i-1,i}$ when $t = T_{i-1}$. Since both numerator and denominator are traded assets, the $L_{t,i}$ is a martingale under $\mathbb{Q}^i$ associated with numeraire $P_{t,i}$

$$L_{t,i} = \frac{P_{t,i-1} - P_{t,i}}{P_{t,i}\tau_i} = \mathbb{E}_t^i\left[\frac{1 - P_{i-1,i}}{P_{i-1,i}\tau_i}\right] = \mathbb{E}_t^i\left[L_{i-1,i}\right] \tag{75}$$

An interest rate swap (IRS) is a contract that exchanges payments between two different (e.g., fixed/floating) interest rate payment legs. At every instant $T_i\ \ \forall\ i = a+1, \cdots, b$, the fixed leg pays out the amount $\tau_i S$ corresponding to a fixed rate S, whereas the floating leg pays the amount $\tau_i L_{i-1,i}$ corresponding to an interest rate $L_{i-1,i}$, e.g., the i-th period spot Libor fixed at T_{i-1}. When the fixed leg is paid and the floating leg is received, the IRS is called Payer IRS, whereas in the other case it is called Receiver IRS. For simplicity, we have assumed that the same schedule applies to both floating leg and fixed leg of the swap. We also disregarded the difference in schedule definitions for interest accrual and for rate fixing. However, proper implementation must take these into account. It has been discussed in detail in chapter 3.

Given an IRS spanning a period from T_a to T_b (i.e., the first value date and the last maturity date, respectively), the swap rate becomes at par if it makes the present value of the fixed leg and the floating leg equal, such that

$$S_t^{a,b} \sum_{i=a+1}^{b} P_{t,i}\tau_i = \sum_{i=a+1}^{b} P_{t,i}\tau_i L_{t,i} = P_{t,a} - P_{t,b} \tag{76}$$

where the par swap rate can be calculated as

$$S_t^{a,b} = \frac{P_{t,a} - P_{t,b}}{\sum_{i=a+1}^{b} \tau_i P_{t,i}} = \frac{1 - \prod_{j=a+1}^{b} \frac{1}{1 + \tau_j L_{t,j}}}{\sum_{i=a+1}^{b} \tau_i \prod_{j=a+1}^{i} \frac{1}{1 + \tau_j L_{t,j}}} \tag{77}$$

Note that if present time $t < T_a$, the swap is forward starting, whereas if $t = T_a$, it becomes a spot starting swap.

4.1. Caps and Floors

Caps/floors and swaptions are two main OTC derivative products in the interest rate markets. The caps and floors at time t are baskets of European calls (i.e., caplets) and puts (i.e., floorlets) on forward Libor rates for a period from T_a to T_b. If $t < T_a$ (here we ignore the spot lag), it is called forward start cap/floor, whereas if $t = T_a$ it is called spot start cap/floor. For example, a 10-year spot start ($t = T_a$) cap struck at K consists of 39 caplets each of which expires at the beginning of each 3M rate period of today's date. The first 3M period $T_a \sim T_{a+1}$ is excluded from the cap because the spot Libor rate $L_{a,a+1}$ is already known and fixed. The cap buyer receives payment $\tau_i \left(L_{i-1,i} - K\right)^+$ at the end of each rate period (except for the first period for a spot cap). According to risk neutral pricing theorem in (8), the cap price is the expected discounted payoffs under $\mathbb{Q}$

$$
\begin{aligned}
V_{t,a,b}^{\text{CAP}} &= \widetilde{\mathbb{E}}_t\left[M_t \sum_{i=a+k}^{b} \frac{1}{M_i}\tau_i\left(L_{i-1,i}-K\right)^+\right], \qquad k=\begin{cases}1 & \text{if } t<T_a\,, \text{ forward cap}\\ 2 & \text{if } t=T_a\,, \text{ spot cap}\end{cases}\\
&= \sum_{i=a+k}^{b}\widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_i}\tau_i\left(L_{i-1,i}-K\right)^+\right] = \sum_{i=a+k}^{b}\mathbb{E}_t^i\left[\frac{P_{t,i}}{P_{i,i}}\tau_i\left(L_{i-1,i}-K\right)^+\right]\\
&= \sum_{i=a+k}^{b} P_{t,i}\tau_i\mathbb{E}_t^i\left[\left(L_{i-1,i}-K\right)^+\right]
\end{aligned}
\tag{78}
$$

where we have changed the numeraire, according to (24), from a money market account M_t to a zero coupon bond $P_{t,i}$ maturing at T_i. The market standard for quoting prices on caps/floors is in terms of *Black's model*, a variant of the Black-Scholes model adapted to handle forward underlying assets. Because, as previously mentioned, $L_{t,i}$ is a martingale under the T_i-forward measure $\mathbb{Q}^i$, it is assumed that $L_{t,i}$ follows a driftless geometric Brownian motion under $\mathbb{Q}^i$ with a deterministic instantaneous volatility $\sigma_{t,i}$, that is

$$
dL_{t,i} = L_{t,i}\sigma_{t,i}dW_t^i \tag{79}
$$

In general, the market quotes cap (floor) prices in terms of *spot* start caps (floors). Hence, the cap price at t can be computed by the following sum of *Black* formulas, each for a caplet

$$
V_{t,a,b}^{\text{CAP}} = \sum_{i=a+2}^{b} P_{t,i}\tau_i\mathfrak{B}\left(K, L_{t,i}, v_i, 1\right) \quad \text{for} \quad t=T_a \tag{80}
$$

where the (undiscounted) *Black* formula $\mathfrak{B}(\cdot)$ is defined as

$$
\begin{aligned}
&\mathfrak{B}(K,F,v,\omega) = \omega F\Phi(\omega d^+) - \omega K\Phi(\omega d^-)\\
&d^+ = \frac{1}{\sqrt{v}}\log\frac{F}{K}+\frac{\sqrt{v}}{2} \qquad \text{and} \qquad d^- = \frac{1}{\sqrt{v}}\log\frac{F}{K}-\frac{\sqrt{v}}{2}
\end{aligned}
\tag{81}
$$

with $\Phi(\cdot)$ denoting the standard normal cumulative density function and $\omega \in \{1,-1\}$ indicating a call or a put. The v_i is the total variance of T_{i-1}-expiry caplet defined as

$$
v_i = \int_t^{T_{i-1}} \sigma_{u,i}^2 du \tag{82}
$$

and the caplet volatility is given by $\sigma_i = \sqrt{\frac{v_i}{T_{i-1}-t}}$.

An spot cap/floor maturing at T_b is said to be at-the-money (ATM) if strike $K = S_{t=T_a}^{a+1,b}$, which is the forward swap rate implied from a zero curve for a period from T_{a+1} to T_b (while $t = T_a$). For simplicity, the market convention is to quote cap/floor volatility in a single number, a flat volatility $\bar{\sigma}_b$. This is the single volatility which, when substituted into the valuation formula for all caplets/floorlets, reproduces the correct market price of the instrument, that is

$$\sum_{i=a+2}^{b} P_{t,i}\tau_i\mathfrak{B}\big(K, L_{t,i}, \bar{\sigma}_b^2(T_{i-1}-t), 1\big) = \sum_{i=a+2}^{b} P_{t,i}\tau_i\mathfrak{B}\big(K, L_{t,i}, v_i, 1\big) \tag{83}$$

Clearly, flat volatility is a dubious concept: since a single caplet may be part of different caps it gets assigned different flat volatilities. The process of constructing implied caplet volatilities from market cap quotes will be discussed as follows.

The market convention quotes the cap/floor price in Black flat volatilities $\bar{\sigma}_b$ where the cap maturity T_b usually takes integer value of years from 1 year up to 30 years. Let us assume the present time $t = T_a$ for spot caps, and we have quarterly resets, so the effective date of the caps is $T_{a+1} = 3M$ and for a 1 year cap its payments are made at times, $T_{a+2} = 6M$, $T_{a+3} = 9M$ and $T_{a+4} = 1Y$, respectively. In order to bootstrap the caplet volatilities for the periods shorter than one year, we need to make some assumptions. We generate two additional caps covering the periods $T_{a+1} \sim T_{a+2}$ and $T_{a+1} \sim T_{a+3}$. Suppose that the volatilities are ATM, the strike prices for these caps equal to the appropriate forward swap rates, $K_b = S_{t=T_a}^{a+1,b}\ \ \forall\ b = a+2, a+3, a+4$, which can be obtained directly from a zero curve. However, we have no cap volatilities for the two additional caps. To obtain these values, one can use constant extrapolation (or any other appropriate extrapolation method). So we may assume that $\bar{\sigma}_{a+2} = \bar{\sigma}_{a+3} = \bar{\sigma}_{a+4}$ where $\bar{\sigma}_{a+4}$ is known and is the 1 year cap volatility. For the broken periods greater than one year (e.g., $T_5 = 1Y3M$) we will be obliged to interpolate (usually using linear interpolation) the market quotes for cap volatilities. Once the whole cap volatility term structure is recovered, we are ready to bootstrap the caplet volatilities [5].

Recall that the spot cap price can be computed by (80)

$$V_{t,a,b}^{\text{CAP}} = \sum_{i=a+2}^{b} P_{t,i}\tau_i\mathfrak{B}\big(K_b, L_{t,i}, \bar{\sigma}_b^2(T_{i-1}-t), 1\big) \tag{84}$$

The bootstrapping is merely using the following equation to recursively compute the caplet volatilities starting from $b = a + 2$

$$\begin{aligned} &P_{t,b}\tau_b\mathfrak{B}\big(K_b, L_{t,b}, v_b, 1\big) \\ &\qquad = \sum_{i=a+2}^{b} P_{t,i}\tau_i\mathfrak{B}\big(K_b, L_{t,i}, \bar{\sigma}_b^2(T_{i-1}-t), 1\big) - \sum_{i=a+2}^{b-1} P_{t,i}\tau_i\mathfrak{B}\big(K_b, L_{t,i}, v_i, 1\big) \end{aligned} \tag{85}$$

For example, we begin with $b = a + 2$ to compute the v_{a+2} of the caplet for rate $L_{t,a+2}$, we have

$$P_{t,a+2}\tau_{a+2}\mathfrak{B}\big(K_{a+2}, L_{t,a+2}, v_{a+2}, 1\big) = P_{t,a+2}\tau_{a+2}\mathfrak{B}\big(K_{a+2}, L_{t,a+2}, \bar{\sigma}_{a+2}^2\sqrt{T_{a+1}-t}, 1\big) \tag{86}$$

This gives $v_{a+2} = \bar{\sigma}_{a+2}^2\sqrt{T_{a+1}-t}$. Given the calculated v_{a+2}, we can further find v_{a+3} for rate $L_{t,a+3}$ such that the following equation holds

$$\begin{aligned} &P_{t,a+3}\tau_{a+3}\mathfrak{B}\big(K_{a+3}, L_{t,a+3}, v_{a+3}, 1\big) \\ &= \sum_{i=a+2}^{a+3} P_{t,i}\tau_i\mathfrak{B}\big(K_{a+3}, L_{t,i}, \bar{\sigma}_{a+3}^2(T_{i-1}-t), 1\big) - P_{t,a+2}\tau_{a+2}\mathfrak{B}\big(K_{a+3}, L_{t,a+2}, v_{a+2}, 1\big) \end{aligned} \tag{87}$$

A univariate nonlinear equation solver is needed to solve (87) for v_{a+3}. Given that v_{a+2} and v_{a+3} are calculated, we are able to uncover v_{a+4} for rate $L_{t,a+4}$, and so on. By repeating the procedure recursively, we will be able to recover all the caplet total variance $v_i \;\forall\, i = a + 2, \cdots, b$ (and therefore the caplet volatilities).

4.2. Swaptions

Interest rate (European) swaptions are options on a payer/receiver IRS (called payer/receiver swaption, respectively). Usually the swaption maturity T coincides with the first reset/fixing date of the underlying IRS (ignoring the spot lag). The underlying IRS length, say from T_a to T_b, is called the tenor of the swaption. The set of reset/fixing and payment dates of the underlying IRS is sometimes called the tenor structure. For example, a $1Y \to 5Y$ (“1 into 5”) payer swaption with strike K gives the holder the right to pay a fixed rate K on a 5 year swap starting in 1 year. A payer swaption is either cash settled or swap settled at its first reset date T_a of the IRS, which is also assumed to be the swaption maturity date. The payer swaption value (e.g., swap settled) at time t is therefore the expected discounted payoffs under risk neutral measure $\mathbb{Q}$

$$\begin{aligned} V_{t,T,a,b}^{\mathrm{PS}} &= \tilde{\mathbb{E}}_t\left[\frac{M_t}{M_T}\left(\sum_{i=a+1}^{b} P_{T,i}\tau_i\big(L_{T,i}-K\big)\right)^+\right] = \tilde{\mathbb{E}}_t\left[\frac{M_t}{M_T}\left(S_T^{a,b}-K\right)^+\sum_{i=a+1}^{b} P_{T,i}\tau_i\right] \\ &= \tilde{\mathbb{E}}_t\left[\frac{M_t}{M_T}\left(S_T^{a,b}-K\right)^+ A_a^{a,b}\right] = \mathbb{E}_t^{a,b}\left[\frac{A_t^{a,b}}{A_T^{a,b}}\left(S_T^{a,b}-K\right)^+ A_T^{a,b}\right] = A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b}-K\right)^+\right] \end{aligned} \tag{88}$$

where we have changed the numeraire from the money market account M_t to the

$$A_t^{a,b} = \sum_{i=a+1}^{b}\tau_i P_{t,i} = \sum_{i=a+1}^{b}\frac{\tau_i}{\prod_{j=a+1}^{i}\big(1+\tau_j L_{t,j}\big)} \tag{89}$$

which is called *annuity*. Since the forward swap rate $S_t^{a,b}$ is given by a market tradable asset $\left(P_{t,a}-P_{t,b}\right)$ denominated in a numeraire $A_t^{a,b}$, it is a martingale under a measure $\mathbb{Q}^{a,b}$ (called swap measure) associated with numeraire $A_t^{a,b}$. Similarly it is assumed that $S_t^{a,b}$ takes a driftless geometric Brownian motion under $\mathbb{Q}^{a,b}$ with a deterministic instantaneous volatility $\sigma_t^{a,b}$

$$dS_t^{a,b} = S_t^{a,b}\sigma_t^{a,b}dW_t^{a,b} \tag{90}$$

This model is known as the swap market model (SMM). The payer swaption (PS) can then be priced by the *Black* formula (81)

$$V_{t,T,a,b}^{\mathrm{PS}} = A_t^{a,b}\mathfrak{B}\big(K,S_t^{a,b},v_{t,T}^{a,b},1\big) \tag{91}$$

where $v_{a,b}$ is the swaption total variance that relates to the instantaneous volatility of $S_t^{a,b}$ by

$$v_{t,T}^{a,b} = \int_t^T \left(\sigma_u^{a,b}\right)^2 du \tag{92}$$

Again, a swaption is said to be ATM if its strike $K = S_t^{a,b}$.

Fundamental difference between the two main interest rate derivatives is that the payoff of swaptions cannot be decomposed into more elementary products. Terminal correlation between different rates can be fundamental in determining swaption price. The term "terminal" is used to stress the correlation that is between the rates (e.g., $L_{i-1,i}$ and $L_{j-1,j}$) rather than between infinitesimal changes in rates (e.g., $dL_{t,i}$ and $dL_{t,j}$). Indeed, we can see from (78) that caps can be decomposed into a sum of the underlying caplets, each depending on a single forward rate along with its marginal distribution. The joint distribution of the rates however is not involved.

5. CONVEXITY ADJUSTMENT

In interest rate modeling, convexity adjustment generally refers to a correction made to the expectation of a stochastic process taken under different probability measures. This correction originates from the extra drift as shown in (32) due to the change of measure (with the associated numeraire). It can be seen that the extra drift is nothing more than a covariance of the underlying stochastic processes. In the following, we are going to introduce three examples of convexity adjustment: 1) Eurodollar Futures 2) Libor-in-arrears and 3) CMS Swap Rates [6].

5.1. Eurodollar Futures

Previously we have briefly discussed the distinction between the FRA rate and Eurodollar futures rate, which originates from the different settlement mechanism. Suppose there is a contract that pays a cashflow of spot Libor $L_{i-1,i}$ at T_i. According to (24), we have the following two martingales:

$$\frac{V_t}{P_{t,i}} = \mathbb{E}_t^i\left[\frac{L_{i-1,i}}{P_{i,i}}\right] = \mathbb{E}_t^i\left[L_{i-1,i}\right] \qquad \text{and} \qquad \frac{V_t}{M_t} = \widetilde{\mathbb{E}}_t\left[\frac{L_{i-1,i}}{M_i}\right] \tag{93}$$

under the T_i-forward measure and risk neutral measure respectively. We already know that the FRA rate $L_{t,i} = \mathbb{E}_t^i\left[L_{i-1,i}\right]$ and the Eurodollar futures rate $\tilde{L}_{t,i} = \widetilde{\mathbb{E}}_t\left[L_{i-1,i}\right]$, the convexity adjustment between the two rates can be derived as follows

$$\begin{aligned} L_{t,i} &= \frac{M_t}{P_{t,i}}\widetilde{\mathbb{E}}_t\left[\frac{L_{i-1,i}}{M_i}\right] = \widetilde{\mathbb{E}}_t\left[L_{i-1,i}\frac{M_t}{P_{t,i}M_i}\right] = \widetilde{\mathbb{E}}_t\left[L_{i-1,i}\frac{D_{t,i}}{P_{t,i}}\right] \\ &= \widetilde{\mathbb{E}}_t\left[L_{i-1,i}\right] + \frac{\widetilde{\mathbb{E}}_t\left[L_{i-1,i}\left(D_{t,i} - P_{t,i}\right)\right]}{P_{t,i}} = \tilde{L}_{t,i} + \frac{\widetilde{\mathbb{E}}_t\left[L_{i-1,i}D_{t,i}\right] - \widetilde{\mathbb{E}}_t\left[L_{i-1,i}\right]\widetilde{\mathbb{E}}_t\left[D_{t,i}\right]}{P_{t,i}} \\ &= \tilde{L}_{t,i} + \frac{\widetilde{\mathbb{V}}_t\left[L_{i-1,i}, D_{t,i}\right]}{P_{t,i}} \end{aligned} \tag{94}$$

This result is consistent with the conclusion in (38). Since $L_{i-1,i}$ and $D_{t,i}$ are usually negatively correlated, the Eurodollar futures rate is slightly higher than the FRA rate.

5.2. Libor-in-Arrears

Suppose there is a contract pays spot Libor rate $L_{i,i+1}$ at the start date of the accrual period T_i rather than at its end date T_{i+1}. Assuming the cashflow has a present value of V_t, according to (24) we have the following martingales

$$\frac{V_t}{P_{t,i}} = \mathbb{E}_t^i\left[\frac{L_{i,i+1}}{P_{i,i}}\right] = \mathbb{E}_t^i\left[L_{i,i+1}\right] \quad \text{and} \quad \frac{V_t}{P_{t,i+1}} = \mathbb{E}_t^{i+1}\left[\frac{L_{i,i+1}}{P_{i,i+1}}\right] \tag{95}$$

under the T_i-forward and T_{i+1}-forward measure respectively. Let us define the forward Libor-in-arrears rate $\mathcal{L}_{t,i+1} = \mathbb{E}_t^i\left[L_{i,i+1}\right]$. This rate is not a martingale under T_i-forward measure and differs from the forward Libor rate $L_{t,i+1} = \mathbb{E}_t^{i+1}\left[L_{i,i+1}\right]$, where a convexity adjustment is needed to amend the gap. From (95) we can easily derive

$$\begin{aligned}
\mathcal{L}_{t,i+1} &= \frac{P_{t,i+1}}{P_{t,i}}\mathbb{E}_t^{i+1}\left[\frac{L_{i,i+1}}{P_{i,i+1}}\right] = \mathbb{E}_t^{i+1}\left[\frac{L_{i,i+1}P_{t,i,i+1}}{P_{i,i+1}}\right] \\
&= \mathbb{E}_t^{i+1}\left[L_{i,i+1}\right] + \mathbb{E}_t^{i+1}\left[\frac{L_{i,i+1}\left(P_{t,i,i+1} - P_{i,i+1}\right)}{P_{i,i+1}}\right] \\
&= L_{t,i+1} + \frac{P_{t,i}}{P_{t,i+1}}\mathbb{E}_t^i\left[\frac{L_{i,i+1}\left(P_{t,i,i+1} - P_{i,i+1}\right)}{P_{i,i}}\right] = L_{t,i+1} + \frac{\mathbb{E}_t^i\left[L_{i,i+1}\left(P_{t,i,i+1} - P_{i,i+1}\right)\right]}{P_{t,i,i+1}} \\
&= L_{t,i+1} - \frac{\mathbb{E}_t^i\left[L_{i,i+1}P_{i,i+1}\right] - \mathbb{E}_t^i\left[L_{i,i+1}\right]\mathbb{E}_t^i\left[P_{i,i+1}\right]}{P_{t,i,i+1}} = L_{t,i+1} - \frac{\mathbb{V}_t^i\left[L_{i,i+1}, P_{i,i+1}\right]}{P_{t,i,i+1}}
\end{aligned} \tag{96}$$

where we have used the following martingale

$$P_{t,i,i+1} = \frac{P_{t,i+1}}{P_{t,i}} = \mathbb{E}_t^i\left[\frac{P_{i,i+1}}{P_{i,i}}\right] = \mathbb{E}_t^i\left[P_{i,i+1}\right] \tag{97}$$

To evaluate the convexity adjustment, we assume the Libor rate follows Black's model with a constant volatility

$$dL_{t,i+1} = L_{t,i+1}\sigma_{i+1}dW_t^{i+1} \tag{98}$$

for which we have the solution given start time s

$$L_{i,i+1} = L_{s,i+1}\exp\left(\sigma_{i+1}W_i^{i+1} - \frac{1}{2}\sigma_{i+1}^2 T_i\right) \tag{99}$$

Since $\mathbb{E}[\exp(\sigma W_T)] = \exp\left(\frac{1}{2}\sigma^2 T\right)$, we can calculate the forward Libor-in-arrears rate as

$$\begin{aligned}\mathcal{L}_{s,i+1} &= \frac{P_{s,i+1}}{P_{s,i}}\mathbb{E}_s^{i+1}\left[\frac{L_{i,i+1}}{P_{i,i+1}}\right] = \frac{\mathbb{E}_s^{i+1}\left[L_{i,i+1}\left(1+\tau_{i+1}L_{i,i+1}\right)\right]}{1+\tau_{i+1}L_{s,i+1}} = \frac{L_{s,i+1}+\tau_{i+1}\mathbb{E}_s^{i+1}\left[L_{i,i+1}^2\right]}{1+\tau_{i+1}L_{s,i+1}} \\ &= \frac{L_{s,i+1}+\tau_{i+1}L_{s,i+1}^2\exp(\sigma_{i+1}^2T_i)}{1+\tau_{i+1}L_{s,i+1}} = L_{s,i+1} + \frac{\tau_{i+1}L_{s,i+1}^2}{1+\tau_{i+1}L_{s,i+1}}[\exp(\sigma_{i+1}^2T_i)-1]\end{aligned} \tag{100}$$

The convexity adjustment is therefore given as follows

$$\mathcal{L}_{s,i+1} - L_{s,i+1} = \frac{\tau_{i+1}L_{s,i+1}^2}{1+\tau_{i+1}L_{s,i+1}}[\exp(\sigma_{i+1}^2T_i)-1] = L_{s,i+1}\theta[\exp(\sigma_{i+1}^2T_i)-1] \tag{101}$$

where

$$\theta = \frac{\tau_{i+1}L_{s,i+1}}{1+\tau_{i+1}L_{s,i+1}} \tag{102}$$

Taking first order expansion, we can approximate (102) by

$$\mathcal{L}_{s,i+1} - L_{s,i+1} \approx L_{s,i+1}\theta\sigma_{i+1}^2T_i \tag{103}$$

5.3. Constant Maturity Swap

The acronym CMS stands for constant maturity swap, which refers to a future fixing of a swap rate. CMS rates are different from the corresponding forward swap rates. CMS rates provide a convenient alternative to Libor as a floating index, as they allow market participants express their views on the future levels of long term rates (for example, the 10 year swap rate).

CMS swaps are commonly structured as Libor for CMS swap. For example, in a Libor for CMS swap, one leg pays a floating coupon indexed by a reference swap rate (e.g., the 10Y swap rate), which fixes two business days before the start of each accrual period. The payments are quarterly on the Act/360 basis and are made at the end of each accrual period. The other leg pays a floating coupon equal to the 3M Libor rate plus a fixed spread, quarterly, on the Act/360 basis. In some cases, the Libor leg of the swap can also be replaced by a fixed rate or potentially another constant maturity rate.

Using a swap rate as the floating rate makes this transaction a bit more difficult to price than a usual Libor based swap. Let us start with a single period CMS swap (i.e., a CMS swaplet) which pays a swap rate $S_a^{a,b}$ at T_p for an accrual period from T_a to T_p, where the dates definition is given as follows

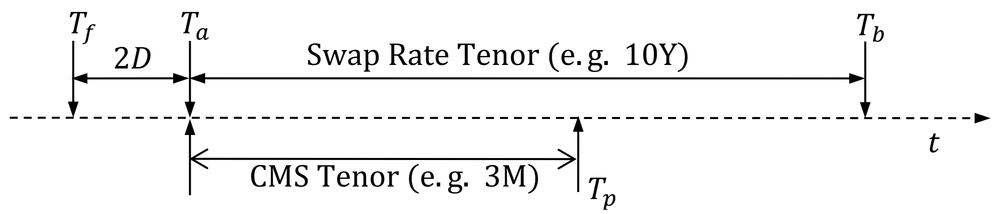


Figure 5.1 One coupon period of a typical CMS leg

In the above diagram, T_a denotes the start date of the reference swap (e.g., 1 year from now). This will also be the start of the accrual period of the CMS swaplet. T_b denotes the maturity date of the reference swap (e.g., 10 years from T_a). T_p denotes the payment day of the CMS swaplet (e.g., 3 months after T_a). This will also be the end of the accrual period of the CMS swaplet. In the name of completeness we should mention that one more date plays a role, namely the date (i.e., fixing date T_f) on which the swap rate is fixed. This is usually two days (i.e., spot lag) before the start date T_a, but in our example we shall neglect its impact.

Given that the payment of $S_a^{a,b}$ amount is made at T_p, according to (24) we would have the following martingales under risk neutral measure and T_p-forward measure, respectively

$$\frac{V_t}{M_t} = \widetilde{\mathbb{E}}_t\left[\frac{S_a^{a,b}}{M_p}\right] \quad \text{and} \quad \frac{V_t}{P_{t,p}} = \mathbb{E}_t^p\left[\frac{S_a^{a,b}}{P_{p,p}}\right] = \mathbb{E}_t^p\left[S_a^{a,b}\right] \tag{104}$$

Since the swap rate $S_a^{a,b}$ is fixed at T_a, we may rewrite the first martingale as

$$\frac{V_t}{M_t} = \widetilde{\mathbb{E}}_t\left[\frac{S_a^{a,b}}{M_p}\right] = \widetilde{\mathbb{E}}_t\left[\widetilde{\mathbb{E}}_a\left[\frac{S_a^{a,b}}{M_p}\right]\right] = \widetilde{\mathbb{E}}_t\left[\frac{S_a^{a,b}}{M_a}\widetilde{\mathbb{E}}_a\left[\frac{M_a}{M_p}\right]\right] = \widetilde{\mathbb{E}}_t\left[\frac{S_a^{a,b}P_{a,p}}{M_a}\right] \tag{105}$$

Note that in (105), the quantity $S_a^{a,b}P_{a,p}$ can be regarded as a cashflow at time T_a, which is equivalent, under risk neutral measure, to a cashflow of $S_a^{a,b}$ at time T_p, providing that the $S_a^{a,b}$ has been fixed at T_a.

After changing numeraire (by (24)) from money market account M_t to annuity $A_t^{a,b}$ (as in (89)), the (105) becomes

$$\frac{V_t}{A_t^{a,b}} = \mathbb{E}_t^{a,b}\left[\frac{S_a^{a,b} P_{a,p}}{A_a^{a,b}}\right] \tag{106}$$

Let us define the forward CMS rate $\mathcal{S}_t^{a,b} \equiv \mathbb{E}_t^p\left[S_a^{a,b}\right]$, which obviously differs from the forward swap rate $S_t^{a,b} = \mathbb{E}_t^{a,b}\left[S_a^{a,b}\right]$. From (104) and (106) we can derive that

$$\begin{aligned}\mathcal{S}_t^{a,b} &= \mathbb{E}_t^p\left[S_a^{a,b}\right] = \mathbb{E}_t^{a,b}\left[S_a^{a,b}\frac{A_t^{a,b}}{A_a^{a,b}}\frac{P_{a,p}}{P_{t,p}}\right] = \mathbb{E}_t^{a,b}\left[S_a^{a,b}\right] + \mathbb{E}_t^{a,b}\left[S_a^{a,b}\left(\frac{A_t^{a,b}}{A_a^{a,b}}\frac{P_{a,p}}{P_{t,p}} - 1\right)\right] \\ &= S_t^{a,b} + \mathbb{E}_t^{a,b}\left[S_a^{a,b}\left(\frac{A_t^{a,b}}{A_a^{a,b}}\frac{P_{a,p}}{P_{t,p}} - 1\right)\right]\end{aligned} \tag{107}$$

The extra term in (107) is the CMS rate convexity adjustment, which can be decomposed into two parts

$$\begin{aligned}&\mathbb{E}_t^{a,b}\left[S_a^{a,b}\left(\frac{A_t^{a,b}}{A_a^{a,b}}\frac{P_{a,p}}{P_{t,p}} - 1\right)\right] = \frac{1}{P_{t,p}}\mathbb{E}_t^{a,b}\left[\frac{S_a^{a,b}\left(A_t^{a,b}P_{a,p} - A_a^{a,b}P_{t,p}\right)}{A_a^{a,b}}\right] \\ &= \frac{1}{P_{t,p}}\frac{P_{t,a}}{A_t^{a,b}}\mathbb{E}_t^a\left[S_a^{a,b}\left(A_t^{a,b}P_{a,p} - A_a^{a,b}P_{t,p}\right)\right] = \frac{\mathbb{E}_t^a\left[S_a^{a,b}P_{a,p}\right]}{P_{t,a,p}} - \frac{\mathbb{E}_t^a\left[S_a^{a,b}A_a^{a,b}\right]}{A_{t,a}^{a,b}} \\ &= \frac{\mathbb{E}_t^a\left[S_a^{a,b}P_{a,p}\right] - \mathbb{E}_t^a\left[S_a^{a,b}\right]\mathbb{E}_t^a\left[P_{a,p}\right]}{P_{t,a,p}} - \frac{\mathbb{E}_t^a\left[S_a^{a,b}A_a^{a,b}\right] - \mathbb{E}_t^a\left[S_a^{a,b}\right]\mathbb{E}_t^a\left[A_a^{a,b}\right]}{A_{t,a}^{a,b}} \\ &= \frac{\mathbb{V}_t^a\left[S_a^{a,b}P_{a,p}\right]}{P_{t,a,p}} - \frac{\mathbb{V}_t^a\left[S_a^{a,b}A_a^{a,b}\right]}{A_{t,a}^{a,b}}\end{aligned} \tag{108}$$

where the following two martingales have been used

$$P_{t,a,p} = \frac{P_{t,p}}{P_{t,a}} = \mathbb{E}_t^a\left[\frac{P_{a,p}}{P_{a,a}}\right] = \mathbb{E}_t^a\left[P_{a,p}\right] \quad \text{and} \quad A_{t,a}^{a,b} = \frac{A_t^{a,b}}{P_{t,a}} = \mathbb{E}_t^a\left[\frac{A_a^{a,b}}{P_{a,a}}\right] = \mathbb{E}_t^a\left[A_a^{a,b}\right] \tag{109}$$

The (108) says the convexity correction can be attributed to two sources: 1) covariance due to the payment delay and 2) the covariance between the swap rate and the annuity factor. Note that the first factor vanishes if the CMS rate is paid at T_a, i.e., at the beginning of the accrual period.

5.3.1. Caplet/Floorlet Replication by Swaptions

The CMS caplet/floorlet can be replicated using swaptions at different strikes. Firstly, we consider a CMS caplet. Its payoff at time T for $T \le T_a$, and its value at initial time t are given by

$$V_{T,K}^{\text{CAP}} = P_{T,p}\left(S_T^{a,b} - K\right)^+ \quad \text{and} \quad V_{t,K}^{\text{CAP}} = P_{t,T}\mathbb{E}_t^T\left[P_{T,p}\left(S_T^{a,b} - K\right)^+\right] \tag{110}$$

Similarly, we write a vanilla payer swaption payoff at T and its value at t by

$$V_{T,K}^{\text{PS}} = A_T^{a,b}\left(S_T^{a,b} - K\right)^+ \quad \text{and} \quad V_{t,K}^{\text{PS}} = A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - K\right)^+\right] \tag{111}$$

where $A_t^{a,b}$ is the annuity. So according to (24), the value of CMS caplet can be expressed as

$$\begin{aligned} V_{t,K}^{\text{CAP}} &= P_{t,T}\mathbb{E}_t^T\left[P_{T,p}\left(S_T^{a,b} - K\right)^+\right] = A_t^{a,b}\mathbb{E}_t^{a,b}\left[\frac{P_{T,p}}{A_T^{a,b}}\left(S_T^{a,b} - K\right)^+\right] \\ &= P_{t,p}\mathbb{E}_t^{a,b}\left[\frac{P_{T,p}}{A_T^{a,b}}\frac{A_t^{a,b}}{P_{t,p}}\left(S_T^{a,b} - K\right)^+\right] = g_t V_{t,K}^{\text{PS}} + \underbrace{P_{t,p}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - K\right)^+\left(\frac{g_T}{g_t} - 1\right)\right]}_{\text{Convexity Adjustment}} \end{aligned} \tag{112}$$

where we define variable $g_u = P_{u,p}/A_u^{a,b}$. The g_u can be approximated by $G\left(S_u^{a,b}\right)$, a function of the swap rate $S_u^{a,b}$ for $t \le u \le T$ (Hagan 2003 [7] provides a few ways to construct the function). Hence the 2nd term in (112), i.e., the convexity adjustment (cc), becomes

$$cc = P_{t,p}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - K\right)^+\left(\frac{G\left(S_T^{a,b}\right)}{G\left(S_t^{a,b}\right)} - 1\right)\right] \tag{113}$$

The payoff can be replicated by payer swaptions. For any smooth function f_x with $f_K = 0$ we can write

$$(S-K)^+ f_K' + \int_K^\infty (S-x)^+ f_x'' dx = \begin{cases} f_S & \text{for } S > K \\ 0 & \text{for } S \le K \end{cases} \tag{114}$$

This can be proved by integration by parts as follows

$$\begin{aligned} &(S-K)^+ f_K' + \int_K^\infty (S-x)^+ f_x'' dx = (S-K)^+ f_K' + \int_K^S (S-x)^+ f_x'' dx \\ &= (S-K)^+ f_K' + \int_K^S (S-x)^+ d f_x' = (S-K)^+ f_K' + (S-x)^+ f_x'\big|_{x=K}^S - \int_K^S f_x' d(S-x)^+ \\ &= (S-K)^+ f_K' - (S-K)^+ f_K' + \int_K^S f_x' \Theta(S-x) dx = \int_K^S \Theta(S-x) d f_x \end{aligned} \tag{115}$$

$$= \begin{cases} f_S - f_K = f_S & \text{for } S > K \\ 0 & \text{for } S \le K \end{cases}$$

In fact, there exists a more general formula [8]. Suppose there is a particular payout $A(S_T)$, and suppose that $A(S_T)$ is a smooth function. It is possible to replicate the payout from vanilla options by

$$A(S) = A(\alpha) + A'(\alpha)(S-\alpha) + \int_\alpha^\infty (S-K)^+ A''(K)dK + \int_0^\alpha (K-S)^+ A''(K)dK \tag{116}$$

where α is any conveniently chosen constant. This tells us that we can perfectly replicate a European payout with an infinite portfolio of vanillas, plus a forward contract and a cash payment. The proof is sketched below, again by integration by parts, starting from the right-hand side of (116)

$$\begin{aligned}
&A(\alpha) + A'(\alpha)(S-\alpha) + \int_\alpha^\infty (S-K)^+ A''(K)dK + \int_0^\alpha (K-S)^+ A''(K)dK \\
&= A(\alpha) + A'(\alpha)(S-\alpha) + \int_\alpha^\infty (S-K)^+ dA'(K) + \int_0^\alpha (K-S)^+ dA'(K) \\
&= A(\alpha) + A'(\alpha)(S-\alpha) + (S-K)^+ A'(K)\Big|_{K=\alpha}^\infty - \int_\alpha^\infty A'(K)d(S-K)^+ \\
&\qquad + (K-S)^+ A'(K)\Big|_{K=0}^\alpha - \int_0^\alpha A'(K)d(K-S)^+ \\
&= A(\alpha) + A'(\alpha)(S-\alpha) - (S-\alpha)^+ A'(\alpha) + \int_\alpha^\infty A'(K)\Theta(S-K)dK + (\alpha-S)^+ A'(\alpha) \\
&\qquad - \int_0^\alpha A'(K)\Theta(K-S)dK \\
&= A(\alpha) + A'(\alpha)(S-\alpha) + ((\alpha-S)^+ - (S-\alpha)^+)A'(\alpha) + \int_\alpha^S A'(K)\Theta(S-K)dK \\
&\qquad - \int_S^\alpha A'(K)\Theta(K-S)dK \\
&= A(\alpha) + A'(\alpha)(S-\alpha) + (\alpha-S)A'(\alpha) + \int_\alpha^S A'(K)\Theta(S-K)dK + \int_\alpha^S A'(K)\Theta(K-S)dK \\
&= A(\alpha) + \int_\alpha^S A'(K)\big(\Theta(S-K) + \Theta(K-S)\big)dK = A(\alpha) + \int_\alpha^S A'(K)dK = A(S)
\end{aligned} \tag{117}$$

An example of function $A(S_T)$ would be $\log S_T$ and we can replicate the payoff by

$$\log S_T = \log\alpha + \frac{S_T}{\alpha} - 1 - \int_\alpha^\infty \frac{(S_T-K)^+}{K^2}dK - \int_0^\alpha \frac{(K-S)^+}{K^2}dK \tag{118}$$

Let us choose the function f_x to be

$$\begin{aligned} & f_x = (x-K)\left(\frac{G(x)}{G\left(S_t^{a,b}\right)} - 1\right) \\ & \Longrightarrow f_x' = \frac{G(x)}{G\left(S_t^{a,b}\right)} - 1 + \frac{G'(x)}{G\left(S_t^{a,b}\right)}(x-K), \qquad f_x'' = \frac{G''(x)(x-K) + 2G'(x)}{G\left(S_t^{a,b}\right)} \end{aligned} \tag{119}$$

and substitute (114) into (113), we have

$$\begin{aligned} cc &= P_{t,p}\mathbb{E}_t^{a,b}\left[f\left(S_T^{a,b}\right)\right] = P_{t,p}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b}-K\right)^+ f_K' + \int_K^\infty \left(S_T^{a,b}-x\right)^+ f_x'' dx\right] \\ &= P_{t,p} f_K' \mathbb{E}_t^{a,b}\left[\left(S_T^{a,b}-K\right)^+\right] + P_{t,p}\int_K^\infty f_x'' \mathbb{E}_t^{a,b}\left[\left(S_T^{a,b}-x\right)^+\right] dx \\ &= g_t\left(f_K' V_{t,K}^{\mathrm{PS}} + \int_K^\infty f_x'' V_{t,x}^{\mathrm{PS}} dx\right) \end{aligned} \tag{120}$$

Hence from (112), the CMS caplet value becomes

$$\begin{aligned} V_{t,K}^{\mathrm{CAP}} &= g_t(1+f_K')V_{t,K}^{\mathrm{PS}} + g_t\int_K^\infty f_x'' V_{t,x}^{\mathrm{PS}} dx \\ &= g_t \frac{G(K)}{G\left(S_t^{a,b}\right)} V_{t,K}^{\mathrm{PS}} + g_t \int_K^\infty V_{t,x}^{\mathrm{PS}} \frac{G''(x)(x-K)+2G'(x)}{G\left(S_t^{a,b}\right)} dx \\ &= G(K)V_{t,K}^{\mathrm{PS}} + \int_K^\infty V_{t,x}^{\mathrm{PS}}\big(G''(x)(x-K)+2G'(x)\big)dx \end{aligned} \tag{121}$$

Assuming $G(x)$ is a linear function of x, e.g., taking first order expansion of $G(x)$ around $S_t^{a,b}$

$$G(x) \approx G\left(S_t^{a,b}\right) + G'\left(S_t^{a,b}\right)\left(x - S_t^{a,b}\right) \Longrightarrow G'(x) = G'\left(S_t^{a,b}\right), \qquad G''(x) = 0 \tag{122}$$

we have

$$V_{t,K}^{\mathrm{CAP}} = g_t V_{t,K}^{\mathrm{PS}} + G'\left(S_t^{a,b}\right)\left(\left(K - S_t^{a,b}\right)V_{t,K}^{\mathrm{PS}} + 2\int_K^\infty V_{t,x}^{\mathrm{PS}} dx\right) \tag{123}$$

$$= g_t V_{t,K}^{\mathrm{PS}} + G'\left(S_t^{a,b}\right)\left(\left(K - S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - K\right)^+\right] + A_t^{a,b}\mathbb{E}_t^{a,b}\left[2\int_K^\infty \left(S_T^{a,b} - x\right)^+ dx\right]\right)$$

$$= g_t V_{t,K}^{\mathrm{PS}} + G'\left(S_t^{a,b}\right)\left(A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(K - S_t^{a,b}\right)\left(S_T^{a,b} - K\right)^+\right] + A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(\left(S_T^{a,b} - K\right)^+\right)^2\right]\right)$$

$$= g_t V_{t,K}^{\mathrm{PS}} + G'\left(S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)\left(S_T^{a,b} - K\right)^+\right]$$

Similarly, by defining receiver swaption value $V_{t,K}^{\mathrm{RS}} = A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(K - S_T^{a,b}\right)^+\right]$, we have the CMS floorlet value

$$\begin{aligned} V_{t,K}^{\mathrm{FLR}} &= g_t V_{t,K}^{\mathrm{RS}} + g_t\left(f_x' V_{t,K}^{\mathrm{RS}} - \int_{-\infty}^K f_x'' V_{t,x}^{\mathrm{RS}} dx\right) \\ &= g_t V_{t,K}^{\mathrm{RS}} - G'\left(S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_t^{a,b} - S_T^{a,b}\right)\left(K - S_T^{a,b}\right)^+\right] \end{aligned} \tag{124}$$

Using put-call parity, the CMS swaplet value can be computed as

$$\begin{aligned} V_{t,K}^{\mathrm{SWP}} &= V_{t,K}^{\mathrm{CAP}} - V_{t,K}^{\mathrm{FLR}} = P_{t,p}\left(S_t^{a,b} - K\right) + G'\left(S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)\left(S_T^{a,b} - K\right)\right] \\ &= P_{t,p}\left(S_t^{a,b} - K\right) + G'\left(S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)^2\right] \end{aligned} \tag{125}$$

where we have used the fact that $S_t^{a,b} = \mathbb{E}_t^{a,b}\left[S_T^{a,b}\right]$. When $K = 0$, it reduces to

$$\begin{aligned} V_t^{\mathrm{SWP}} &= P_{t,p}S_t^{a,b} + G'\left(S_t^{a,b}\right)A_t^{a,b}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)^2\right] \\ \Longrightarrow \mathcal{S}_t^{a,b} &= S_t^{a,b} + \frac{G'\left(S_t^{a,b}\right)}{G\left(S_t^{a,b}\right)}\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)^2\right] \end{aligned} \tag{126}$$

Note that by assuming $G(x)$ is a linear function of x, the convexity adjustment is determined by the term $\mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_t^{a,b}\right)^2\right]$, i.e., the variance of the $S_T^{a,b}$ under the swap measure with annuity as the numeraire.

5.3.2. Discrete Replication by Swaptions

Another approach is to *statically* replicate the CMS payoff by a discrete portfolio of European swaptions. The idea is to replicate the linear payoff of CMS caplets/floorlets with the concave/convex payoff of European swaptions at different strike prices in such a way that the distance between both payoffs is minimized. Let us first write a CMS caplet payoff as a linear combination (with static weights) of a series of payer swaption payoffs with strikes $K = K_0 < K_1 < \cdots < K_N$

$$V_{T,K}^{\text{CAP}} = P_{T,p}\left(S_T^{a,b} - K\right)^+ = \sum_{h=0}^{N-1} \omega_h A_T^{a,b}\left(S_T^{a,b} - K_h\right)^+ \tag{127}$$

Then the CMS caplet value at t can be written as

$$\begin{aligned} V_{t,K}^{\text{CAP}} &= P_{t,T}\mathbb{E}_t^T\left[P_{T,p}\left(S_T^{a,b} - K\right)^+\right] = P_{t,T}\mathbb{E}_t^T\left[A_T^{a,b}\sum_{h=0}^{N-1}\omega_h\left(S_T^{a,b} - K_h\right)^+\right] \\ &= A_t^{a,b}\mathbb{E}_t^{a,b}\left[\sum_{h=0}^{N-1}\omega_h\left(S_T^{a,b} - K_h\right)^+\right] = \sum_{h=0}^{N-1}\omega_h V_{t,h}^{\text{PS}} \end{aligned} \tag{128}$$

As long as we can calculate the static weights, the replication is trivial.

Again, we approximate the quantity $g_u = P_{u,p}/A_u^{a,b}$ by a function $G\left(S_u^{a,b}\right)$ of the swap rate $S_u^{a,b}$, such that

$$G\left(S_T^{a,b}\right)\left(S_T^{a,b} - K\right)^+ = \sum_{h=0}^{N-1}\omega_h\left(S_T^{a,b} - K_h\right)^+ \tag{129}$$

To derive the weights ω_h, we first let $S_T^{a,b} = K_1$, this gives

$$G(K_1)\,(K_1 - K_0) = \sum_{h=0}^{0}\omega_h(K_1 - K_h)^+ = \omega_0(K_1 - K_0) \Longrightarrow \omega_0 = G(K_1) \tag{130}$$

and then $S_T^{a,b} = K_2$, such that

$$\begin{aligned} &G(K_2)\,(K_2 - K_0) = \sum_{h=0}^{1}\omega_h(K_2 - K_h) = \omega_0(K_2 - K_0) + \omega_1(K_2 - K_1) \\ &\Longrightarrow \omega_1 = \frac{G(K_2)\,(K_2 - K_0) - \omega_0(K_2 - K_0)}{K_2 - K_1} \end{aligned} \tag{131}$$

when $S_T^{a,b} = K_{j+1}\ \forall\, 0 \le j \le N-1$, we have ω_j to be defined recursively

$$G\left(K_{j+1}\right)\left(K_{j+1}-K_0\right)=\sum_{h=0}^{j}\omega_h\left(K_{j+1}-K_h\right)=\omega_j\left(K_{j+1}-K_j\right)+\sum_{h=0}^{j-1}\omega_h\left(K_{j+1}-K_h\right)$$
$$\Longrightarrow \omega_j=\frac{G\left(K_{j+1}\right)\left(K_{j+1}-K_0\right)-\sum_{h=0}^{j-1}\omega_h\left(K_{j+1}-K_h\right)}{K_{j+1}-K_j} \tag{132}$$

In the case of floorlet, we have the replication portfolio of N receiver swaptions at strikes $K=K_0>K_1>\cdots>K_N$

$$V_{t,K}^{\mathrm{FLR}}=P_{t,T}\mathbb{E}_t^T\left[P_{T,p}\left(K-S_T^{a,b}\right)^+\right]=P_{t,T}\mathbb{E}_t^T\left[A_T^{a,b}\sum_{h=0}^{N-1}\omega_h\left(K_h-S_T^{a,b}\right)^+\right]=\sum_{h=0}^{N-1}\omega_h V_{t,h}^{\mathrm{RS}} \tag{133}$$

Writing the payoff

$$G\left(S_T^{a,b}\right)\left(K-S_T^{a,b}\right)^+=\sum_{h=0}^{N-1}\omega_h\left(K_h-S_T^{a,b}\right)^+ \tag{134}$$

the weights can be derived in the same manner. Firstly let $S_T^{a,b}=K_1$, this gives

$$G(K_1)\,(K_0-K_1)=\sum_{h=0}^{0}\omega_h(K_h-K_1)^+=\omega_0(K_0-K_1)\Longrightarrow\omega_0=G(K_1) \tag{135}$$

and then let $S_T^{a,b}=K_2$ to yield

$$G(K_2)\,(K_0-K_2)=\sum_{h=0}^{1}\omega_h(K_h-K_2)=\omega_0(K_0-K_2)+\omega_1(K_1-K_2)$$
$$\Longrightarrow\omega_1=\frac{G(K_2)\,(K_0-K_2)-\omega_0(K_0-K_2)}{K_1-K_2} \tag{136}$$

when $S_T^{a,b}=K_{j+1}\ \forall\ 0\le j\le N-1$, we have

$$G\left(K_{j+1}\right)\left(K_0-K_{j+1}\right)=\sum_{h=0}^{j}\omega_h\left(K_h-K_{j+1}\right)=\omega_j\left(K_j-K_{j+1}\right)+\sum_{h=0}^{j-1}\omega_h\left(K_h-K_{j+1}\right)$$
$$\Longrightarrow\omega_j=\frac{G\left(K_{j+1}\right)\left(K_0-K_{j+1}\right)-\sum_{h=0}^{j-1}\omega_h\left(K_h-K_{j+1}\right)}{K_j-K_{j+1}} \tag{137}$$

In fact, the weights for floorlets retain the same form as for the caplets.

In the following, we will introduce two definitions for the $G\left(S_T^{a,b}\right)$ function on $S_T^{a,b}$.

5.3.2.1. *Linear Swap Rate Model*

In the linear swap rate model, we want to approximate the $G\left(S_T^{a,b}\right)$ by a linear swap rate function

$$\frac{P_{T,p}}{A_T^{a,b}} = G\left(S_T^{a,b}\right) = \alpha S_T^{a,b} + \beta \tag{138}$$

We will know in next chapters that, in the context of Gaussian 1-factor model, the zero coupon bond, the annuity and the swap rate are all functions of a common factor, the short rate r. For example, the zero coupon bond admit an affine term structure, such that $P_{T,p} = \exp\left(-A_{T,p} - B_{T,p} r\right)$, where $B_{T,p} = \int_T^p \exp\left(-\int_T^u \kappa_s ds\right) du$ or $B_{T,p} = \frac{1-e^{-\kappa(p-T)}}{\kappa}$ if mean reversion rate κ is a constant. Hence the slope coefficient α can be calculated as

$$\alpha = \frac{\partial G\left(S_T^{a,b}\right)}{\partial S_T^{a,b}} = \frac{\frac{\partial G\left(S_T^{a,b}\right)}{\partial r}}{\frac{\partial S_T^{a,b}}{\partial r}} = \frac{P_{T,p}}{A_T^{a,b}} \frac{\sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i} - B_{T,p} P_{T,p}}{S_T^{a,b} \sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i} + B_{T,b} P_{T,b} - B_{T,a} P_{T,a}} \tag{139}$$

where the derivatives are as follows

$$\frac{\partial P_{T,p}}{\partial r} = -B_{T,p} P_{T,p} \qquad \text{and} \qquad \frac{\partial A_T^{a,b}}{\partial r} = \frac{\partial}{\partial r} \sum_{i=a+1}^b \tau_i P_{T,i} = -\sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i}$$

$$\begin{aligned} \frac{\partial G\left(S_T^{a,b}\right)}{\partial r} &= \frac{\partial}{\partial r} \frac{P_{T,p}}{A_T^{a,b}} = \frac{1}{A_T^{a,b}} \frac{\partial P_{T,p}}{\partial r} - \frac{P_{T,p}}{\left(A_t^{a,b}\right)^2} \frac{\partial A_T^{a,b}}{\partial r} = -B_{T,p} \frac{P_{T,p}}{A_T^{a,b}} + \frac{P_{T,p}}{\left(A_T^{a,b}\right)^2} \sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i} \\ &= \frac{P_{T,p}}{A_T^{a,b}} \left(\frac{\sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i}}{A_T^{a,b}} - B_{T,p} \right) \end{aligned} \tag{140}$$

$$\frac{\partial S_T^{a,b}}{\partial r} = \frac{\partial}{\partial r} \frac{P_{T,a}}{A_T^{a,b}} - \frac{\partial}{\partial r} \frac{P_{T,b}}{A_T^{a,b}} = \frac{S_T^{a,b}}{A_T^{a,b}} \sum_{i=a+1}^b \tau_i B_{T,i} P_{T,i} - \frac{P_{T,a}}{A_T^{a,b}} B_{T,a} + \frac{P_{T,b}}{A_T^{a,b}} B_{t,b}$$

Using the initial freeze approximation (e.g., $P_{T,p} \approx P_{t,p}$ and $A_T^{a,b} \approx A_t^{a,b}$), we get

$$\alpha \approx \frac{P_{t,p}}{A_t^{a,b}} \frac{\sum_{i=a+1}^b \tau_i B_{T,i} P_{t,i} - B_{T,p} P_{t,p}}{S_t^{a,b} \sum_{i=a+1}^b \tau_i B_{T,i} P_{t,i} + B_{T,b} P_{t,b} - B_{T,a} P_{t,a}} \tag{141}$$

Because $P_{T,p}/A_T^{a,b}$ is a martingale under the swap measure associated with $A_t^{a,b}$, we derive β by

$$\frac{P_{t,p}}{A_t^{a,b}} = \mathbb{E}_t^{a,b}\left[\frac{P_{T,p}}{A_T^{a,b}}\right] = \alpha\mathbb{E}_t^{a,b}\left[S_T^{a,b}\right] + \beta = \alpha S_t^{a,b} + \beta \implies \beta = \frac{P_{t,p}}{A_t^{a,b}} - \alpha S_t^{a,b} \tag{142}$$

In (128) or (133), the CMS caplet or floorlet with strike K is replicated by a portfolio of vanilla payer or receiver swaptions with strikes $K_h \in [K, K_{\max}]$ or $K_h \in [K_{\min}, K]$he lower or upper bound of strikes, $K_{\min}$d $K_{\max}$an be set by

$$K_{\min} = S_t^{a,b} \exp\left(-n\Sigma - \frac{1}{2}\Sigma^2\right), \qquad K_{\max} = S_t^{a,b} \exp\left(n\Sigma - \frac{1}{2}\Sigma^2\right) \tag{143}$$

where $\Sigma = \sigma\sqrt{T-t}$ is the ATM total Black volatility of swap rate $S_T^{a,b}$ and n corresponds to number of standard deviations (e.g., usually $n = 5$). For strikes K_h, the range (e.g., $[K, K_{\max}]$ or $[K_{\min}, K]$an be spaced uniformly or log-uniformly. In fact, one may make it more flexible and use vega to determine the bounds. For example, we may firstly calculate the vega of an ATM swaption on $S_T^{a,b}$, and then find the strikes of the swaption having a vega 100 times smaller (by inverting the vega Black formula. Note that there are two solutions corresponding to lower and upper bound respectively). The strikes found can be used as the bounds.

5.3.2.2. *Hagan Swap Rate Model*

This model takes into account the initial yield curve shape and allows (only) parallel yield curve shifts (see appendix A.3 in Hagan 2003 [7]). Again we assume the dynamics of yield curve follows Gaussian 1-factor short rate model. By assuming constant mean reversion rate κ, we have $B_{t,T} = \frac{1-e^{-\kappa(T-t)}}{\kappa}$. The shifted zero-coupon bond is then given by initial yield curve shape and a shift ξ

$$P_{T,V}(\xi) = P_{t,T,V} \exp\left(-B_{T,V}\xi\right) \qquad \text{and} \qquad P_{t,T,V} = \frac{P_{t,V}}{P_{t,T}} \tag{144}$$

Consequently, we have the annuity, the swap rate and the function G defined as

$$A_T^{a,b}(\xi) = \sum_{i=a+1}^{b} \tau_i P_{T,i}(\xi), \qquad S_T^{a,b}(\xi) = \frac{P_{T,a}(\xi) - P_{T,b}(\xi)}{A_T^{a,b}(\xi)}, \qquad G(\xi) = \frac{P_{T,p}(\xi)}{A_T^{a,b}(\xi)} \tag{145}$$

Since there is a 1-to-1 mapping from ξ to swap rate $S_T^{a,b}$, we may find the corresponding ξ for a given swap rate S by inverting $S = S_T^{a,b}(\xi)$. The range of strike for K_h in (128) and (133) can then be remapped in terms of the shift ξ, e.g., $\xi_h \in [\xi_K, \xi_{\max}]$ for $K_h \in [K, K_{\max}]$. The replication is performed using a series of uniformly spaced ξ_h, that is

$$\begin{aligned} V_{t,K}^{\text{CAP}} &= \sum_{h=0}^{N} \omega_h V_{t,h}^{\text{PS}} \quad \text{and} \quad V_{t,h}^{\text{PS}} = A_t^{a,b} \mathbb{E}_t^{a,b}\left[\left(S_T^{a,b} - S_T^{a,b}(\xi_h)\right)^+\right], \quad \xi_h \in [\xi_K, \xi_{\max}] \\ V_{t,K}^{\text{FLR}} &= \sum_{h=0}^{N} \omega_h V_{t,h}^{\text{RS}} \quad \text{and} \quad V_{t,h}^{\text{RS}} = A_t^{a,b} \mathbb{E}_t^{a,b}\left[\left(S_T^{a,b}(\xi_h) - S_T^{a,b}\right)^+\right], \quad \xi_h \in [\xi_{\min}, \xi_K] \end{aligned} \tag{146}$$

5.3.2.3. *Treatment in Multi-Curve Framework*

As discussed in chapter 3, in the context of multi-curve framework, the swap rate is determined jointly by the projection curve and discounting curve. Namely, we must calculate the swap rate by the formula

$$S_T(\xi) = \frac{1}{\sum_j c_{j,a} P_{T,p_j}(\xi)} \sum_i \frac{\hat{P}_{T,f_{i,s}}(\xi) - \hat{P}_{T,f_{i,e}}(\xi)}{c_{i,f} \hat{P}_{T,f_{i,e}}(\xi)} c_{i,a} P_{T,p_i}(\xi) \tag{147}$$

where we can estimate the projection curve $\hat{P}_{T,V}(\xi) = \hat{P}_{t,T,V} \exp(-B_{T,V}\xi)$ and the discounting curve $P_{T,V}(\xi) = P_{t,T,V} \exp(-B_{T,V}\xi)$ under the assumption of constant multiplicative spread (see section 8.5.2).

6. Finite Difference Methods

For a comprehensive discussion of this topic in the context of derivative pricing, please refer to [9] [10] and [11].

6.1. Partial Differential Equations

Suppose a multi-factor stochastic process is governed by the following SDE

$$d\underset{n\times 1}{x} = \underset{n\times 1}{\mu}\, dt + \underset{n\times n}{\sigma}\, dW_t, \qquad dW_t dW_t' = \underset{n\times n}{\rho}\, dt \tag{148}$$

where we use prime symbol (e.g., W') to denote transpose operation. The vectors are defined as follows

$$\underset{n\times 1}{x} = \begin{bmatrix} \vdots \\ x_{i;t} \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{\mu} = \begin{bmatrix} \vdots \\ \mu_{i;t,x} \\ \vdots \end{bmatrix}, \qquad \underset{n\times n}{\sigma} = \mathrm{Diag}\begin{bmatrix} \vdots \\ \sigma_{i;t,x} \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{W_t} = \begin{bmatrix} \vdots \\ W_{i;t} \\ \vdots \end{bmatrix} \tag{149}$$

The $\sigma = \sigma_{t,x}$ is an $n \times n$ diagonal matrix and $W = W_t$ is an $n \times 1$ correlated Brownian motion (with correlation matrix ρ) under a measure associate with a numeraire N. In the context of derivative pricing, we likely encounter two types of PDEs associated with this stochastic process. The first one is called *Kolmogorov* forward equation (i.e., *Fokker-Planck* equation), which governs the evolution of the transition probability density function $p(t,\beta|s,\alpha)$ of the stochastic process x_t, e.g., the transition density having $x_t = \beta$ at time t given $x_s = \alpha$ at start time s. The second is the backward PDE given by *Feynman-Kac* Theorem, which governs the evolution of derivative value under an equivalent martingale measure.

6.1.1. Kolmogorov Forward Equation

Numerical solution of transition probability density function is often sought for the purpose of model calibration. Its evolution is characterized by Kolmogorov forward equation below

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial(\mu_i p)}{\partial x_i} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\left(\Sigma_{ij} p\right)}{\partial x_i \partial x_j} = 0, \qquad \lim_{t\to s} p_{t,x|s,\alpha} = \delta_{x-\alpha}, \qquad \Sigma_{ij} = \sigma_i \rho_{ij} \sigma_j \tag{150}$$

with $\delta(\cdot)$ the Dirac delta function as its initial condition. It basically tells that if we solve the equation with initial condition at $x_s = \alpha \in \Omega \equiv \mathbb{R}^n$ at time s, then we would be able to find $x_t = \beta \in \Omega$ at time t with a probability density $p(t,\beta|s,\alpha)$.

6.1.2. Evolution of Derivative Price

Suppose asset $U(t,x_t)$ and numeraire $N(t,x_t)$ both are driven by the same stochastic process x_t, the dynamics of U can be expressed as

$$\begin{aligned} dU &= \frac{\partial U}{\partial t}dt + J_U dx + \frac{1}{2}dx' H dx = \frac{\partial U}{\partial t}dt + J_U \mu dt + J_U \sigma dW + \frac{dW'\sigma H_U \sigma dW}{2} \\ &= \left(\frac{\partial U}{\partial t} + J_U\mu + \frac{\mathbb{1}'\sigma(H_U \cdot \rho)\sigma\mathbb{1}}{2}\right)dt + J_U\sigma dW \end{aligned} \tag{151}$$

where the $\mathbb{1}$ denotes an $n \times 1$ all-ones vector used to aggregate vector/matrix elements. The *Jacobian* J_U and the *Hessian* H_U are defined as

$$\underset{1\times n}{J_U} = \frac{\partial U}{\partial x}, \qquad [J_U]_i = \frac{\partial U}{\partial x_i} \qquad \text{and} \qquad \underset{n\times n}{H_U} = \frac{\partial^2 U}{\partial x^2}, \qquad [H_U]_{i,j} = \frac{\partial^2 U}{\partial x_i \partial x_j} \tag{152}$$

and the $H_U \cdot \rho$ denotes element-wise multiplication. Assume the numeraire N is lognormal and its dynamics can be described by

$$\frac{1}{N}dN = \theta dt + \xi' dW, \qquad d\frac{1}{N} = -\frac{1}{N^2}dN + \frac{1}{2}\frac{2}{N^3}dNdN = \frac{1}{N}(\xi'\rho\xi - \theta)dt - \frac{1}{N}\xi' dW \tag{153}$$

where $\theta = \theta(t,x_t)$ is a scalar drift and $\xi = \xi(t,x_t)$ is an $n \times 1$ volatility of N. The N-denominated derivative price $U/N = U(t,x_t)/N(t,x_t)$ driven by x_t possesses dynamics as follows

$$\begin{aligned} &Nd\frac{U}{N} = N\left(\frac{1}{N}dU + Ud\frac{1}{N} + dUd\frac{1}{N}\right) \\ &= \left(\frac{\partial U}{\partial t} + J_U\mu + \frac{\mathbb{1}'\sigma(H_N \cdot \rho)\sigma\mathbb{1}}{2}\right)dt + J_U\sigma dW + U\big((\xi'\rho\xi - \theta)dt - \xi' dW\big) - J_U\sigma\rho\xi dt \\ &= \left(\frac{\partial U}{\partial t} + \frac{\mathbb{1}'\sigma(H_N \cdot \rho)\sigma\mathbb{1}}{2} + J_U(\mu - \sigma\rho\xi) + (\xi'\rho\xi - \theta)U\right)dt + J_U\sigma dW - U\xi' dW \end{aligned} \tag{154}$$

Since the U/N is a martingale under the measure associated with N, the drift term must vanish, which gives the PDE that governs the price process as follows

$$\frac{\partial U}{\partial t} = -\frac{\mathbb{1}'\sigma(H_U \cdot \rho)\sigma\mathbb{1}}{2} + J_U(\sigma\rho\xi - \mu) + (\theta - \xi'\rho\xi)U \qquad \text{or} \tag{155}$$

$$\frac{\partial U}{\partial t} = -\frac{1}{2}\sum_{i,j=1}^{n}\sigma_i\rho_{ij}\sigma_j\frac{\partial^2 U}{\partial x_i \partial x_j} + \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sigma_i\rho_{ij}\xi_j - \mu_i\right)\frac{\partial U}{\partial x_i} + \left(\theta - \sum_{i,j=1}^{n}\xi_i\rho_{ij}\xi_j\right)U$$

The price U evolves backwards in time with a terminal condition known at time $T > t$.

6.1.3. Boundary Conditions

Note that in order to find meaningful solutions to the PDE's, initial or terminal conditions must be specified. In addition, proper boundary conditions must also be provided. Defining $\Gamma(t)$ to be the boundary of the spatial domain, we present below a few types of boundary conditions that are used frequently in practice.

Table 6.1 PDE boundary conditions

Type	Definition
Dirichlet	$U(t,x) = f_D(t,x) \quad \forall\, x \in \Gamma(t)$
Neumann	$\frac{\partial U}{\partial x}(t,x) = f_N(t,x) \quad \forall\, x \in \Gamma(t)$
Convexity	$\frac{\partial^2 U}{\partial x^2}(t,x) = f_C(t,x) \quad \forall\, x \in \Gamma(t)$
Exponential	$\frac{\partial^2 U}{\partial x^2}(t,x) = \frac{\partial U}{\partial x}(t,x) \quad \forall\, x \in \Gamma(t)$

In the table, the f_D, f_N and f_C are some deterministic functions. There is nothing special for the top three boundary conditions. The last one, exponential boundary condition, comes from the fact that most of the time, the diffused variable is the log spot, i.e., $x = \log S$. Since majority of derivative contracts has a payoff (e.g., $(S-K)^+$ for a call) that is ultimately exponential in x (or linear in the underlying spot S), we have $\partial^2 U/\partial S^2 = 0$ at boundary. Namely, the gamma sensitivity is zero. This translates into $\partial^2 U/\partial x^2 = \partial U/\partial x$ with respect to the variable x for $x \in \Gamma(t)$. As such, the exponential boundary condition is also often known as zero gamma boundary condition.

6.2. Finite Difference Solver in One-Dimension

Writing the forward equation (150) in 1D, we get

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2(\sigma^2 p)}{\partial x^2} - \frac{\partial(\mu p)}{\partial x}, \qquad \lim_{t\to s} p_{t,x|s,\alpha} = \delta_{x-\alpha} \tag{156}$$

Note that the partial derivatives are taken on $\sigma^2 p$ and μp rather than just on p, as both μ and σ are dependent on x. This is simpler for the backward equation (155), which in 1D has the form as

$$\frac{\partial U}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} + (\sigma\xi - \mu)\frac{\partial U}{\partial x} + (\theta - \xi^2)U \tag{157}$$

In the backward equation, we may assume money market account as the numeraire, which gives $\theta = r_{t,x}$ and $\xi = 0$, the (157) further reduces to

$$\frac{\partial U}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} - \mu\frac{\partial U}{\partial x} + rU \tag{158}$$

with a terminal condition U_T given by payoff function upon trade maturity. To solve these PDE's, we need first to introduce discretization methods for the spatial and temporal domains.

6.2.1. Non-Uniform Spatial Discretization

We discretize the spatial domain of x to form a computational grid. For a typical lognormal process, the spatial domain is usually defined in log space. In a simple application, a uniform grid with equal spacing is sufficient. However, a non-uniform grid is often in favor in practice with denser distribution of grid points around some critical values, such as spot, strike and barriers. Without loss of generality, we will derive the finite difference approximation of partial derivatives based on a non-uniform grid. The conclusion can be easily extended to the case of a uniform grid. Firstly, we must specify the lower bound L and upper bound H of the domain. These can be estimated, for example, as a few standard deviations away from the spot (or log spot). The bounds can be overridden if a product to be priced has a natural boundary. For instance, the upper barrier level of an up-and-out call option can be used as the upper bound if the barrier is lower than the aforesaid H.

Given boundaries of the domain, there are many ways to generate non-uniform grids. In the following, we will discuss one of the methods. At first, let us define a variable ξ such that $0 \le \xi \le 1$. In discrete case, we can choose $\xi = i/m$ for $i = 0, \cdots, m$ where m is the size of the grid (i.e., there are $n = m + 1$ grid points including the two end points). Based on the uniform grid of ξ, we would like to construct a non-uniform gird $x(\xi)$, which is transformed from ξ with two boundaries defined as $x(0) = L$ and $x(1) = H$.

6.2.1.1. *Grid Generation: Single Critical Value*

Introducing a critical value C such that $L < C < H$, we want to have denser grid point distribution around the value C. This can be achieved by using a Jacobian function defined as

$$J(\xi) = \frac{dx(\xi)}{d\xi} = \sqrt{\alpha^2 + (x(\xi) - C)^2} \tag{159}$$

where α is a prescribed constant controls the grid uniformity. The ordinary differential equation (159) can be solved analytically and considering the boundary values we get the solution [12]

$$x(\xi) = C + \alpha \sinh\left(\xi \operatorname{arcsinh}\frac{H - C}{\alpha} + (1 - \xi)\operatorname{arcsinh}\frac{L - C}{\alpha}\right), \qquad \alpha = \beta(H - L) \tag{160}$$

where β is a constant (e.g., 0.1, the larger the β, the more uniform the grid spacing) and the hyperbolic sine function and its inverse are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \operatorname{arcsinh} x = \log\left(x + \sqrt{x^2 + 1}\right) \tag{161}$$

6.2.1.2. *Grid Generation: Multiple Critical Values*

The transformation (160) can be further generalized to construct grid that takes care of multiple critical values, e.g., $L \le C_i \le H$ for $i = 1, \cdots, k$. In this method, a global Jacobian is defined as

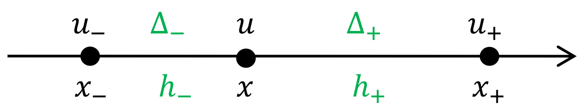
$$J(\xi) = \frac{dx(\xi)}{d\xi} = \lambda\left(\sum_{i=1}^{k} J_i(\xi)^{-2}\right)^{-\frac{1}{2}} \qquad \text{and} \qquad J_i(\xi) = \sqrt{\alpha_i^2 + (x(\xi) - C_i)^2} \tag{162}$$

where the local Jacobian functions $J_i(\xi)$ are in the same form of (159) and λ is a parameter to be determined by boundary values [13] [14] [15]. It is evident that near the critical values the global Jacobian is dominated by the behavior of the local Jacobian, but the influence of nearby critical points ensures that the transitions between them are smooth. In general, the global Jacobian must be integrated numerically (e.g., using Runge-Kutta method) to yield the $x(\xi)$. The numerical integration starts from lower boundary value $x(0) = L$ and adjust the parameter λ such that the upper boundary value $x(1) = H$ is satisfied. Since $x(1)$ is monotonically increasing with λ, the numerical iterations are guaranteed to converge.

Lastly, we usually want to fine tune the non-uniform grid for the sake of numerical accuracy. For example, we usually want the spot fall right onto a node of the grid. We also want to place certain quantities, such as barrier levels and strikes, in the middle of two adjacent grid nodes.

6.2.2. Approximation of Partial Derivatives

Suppose we have the following non-uniform grid for spatial variable x with spacing h



where the simplified notations are defined below

$$\begin{aligned} &u_- = u_{i-1}, \qquad u = u_i, \qquad u_+ = u_{i+1} \\ &x_- = x_{i-1}, \qquad x = x_i, \qquad x_+ = x_{i+1} \\ &h_- = x_i - x_{i-1}, \qquad h_+ = x_{i+1} - x_i \end{aligned} \tag{163}$$

We may also define a time t state price U_t on the grid x, which is a $n \times 1$ vector (for $n = m + 1$) with entries $u_i = U(t, x_i)$

$$\underset{n\times 1}{U_t} = \begin{bmatrix} u_0 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}, \qquad \underset{n\times 1}{x} = \begin{bmatrix} x_0 \\ \vdots \\ x_i \\ \vdots \\ x_m \end{bmatrix} \tag{164}$$

As U_t is a function of time t and spatial variable x. Taylor expansion of U around $x = x_i$ gives

$$\begin{aligned} u_+ &= u + h_+ \frac{\partial U}{\partial x} + \frac{h_+^2}{2}\frac{\partial^2 U}{\partial x^2} + \frac{h_+^3}{6}\frac{\partial^3 U}{\partial x^3} + O(h_+^4) \\ u_- &= u - h_- \frac{\partial U}{\partial x} + \frac{h_-^2}{2}\frac{\partial^2 U}{\partial x^2} - \frac{h_-^3}{6}\frac{\partial^3 U}{\partial x^3} + O(h_-^4) \end{aligned} \tag{165}$$

After simple algebraic deduction, we have

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{h_-}{h_+ + h_-}\frac{u_+ - u}{h_+} + \frac{h_+}{h_+ + h_-}\frac{u - u_-}{h_-} - \frac{h_+ h_-}{6}\frac{\partial^3 U}{\partial x^3} + O(h^3) \\ \frac{\partial^2 U}{\partial x^2} &= \frac{2}{h_+ + h_-}\left(\frac{u_+ - u}{h_+} - \frac{u - u_-}{h_-}\right) - \frac{h_+ - h_-}{3}\frac{\partial^3 U}{\partial x^3} + O(h^3)\end{aligned} \tag{166}$$

Often the non-uniform grid will occur as a transformation of a uniform grid z_i, such that the spacing $h = z_{i+1} - z_i$ remains constant and $x_i = g(z_i)$. As such, the term $h_+ - h_-$ in the above equation admits a quadratic convergence, which can be shown as below

$$h_+ - h_- = g(z_{i+1}) + g(z_{i-1}) - 2g(z_i) = h^2 \frac{d^2 g}{dz^2} + O(h^3) \tag{167}$$

Truncating the higher order terms in (166), the partial derivatives can be approximated by the following finite difference schemes, both are second order accurate

$$\begin{aligned}\frac{\partial U}{\partial x} &\approx -\frac{h_+}{(h_+ + h_-)h_-}u_- + \frac{h_+ - h_-}{h_+ h_-}u + \frac{h_-}{(h_+ + h_-)h_+}u_+ = \Delta_x U \\ \frac{\partial^2 U}{\partial x^2} &\approx \frac{2}{(h_+ + h_-)h_-}u_- - \frac{2}{h_+ h_-}u + \frac{2}{(h_+ + h_-)h_+}u_+ = \Delta_{xx} U\end{aligned} \tag{168}$$

where Δ_x and Δ_{xx} are the first order and second order finite difference operators, respectively. They can be constructed as tridiagonal matrices, with entries as follows

Table 6.2 Formulas for first order and second order differential operators

	sub-diagonal ($j = i - 1$)	diagonal ($j = i$)	super-diagonal ($j = i + 1$)
Δ_x	$-\frac{h_+}{(h_+ + h_-)h_-}$	$\frac{h_+ - h_-}{h_+ h_-}$	$\frac{h_-}{(h_+ + h_-)h_+}$
Δ_{xx}	$\frac{2}{(h_+ + h_-)h_-}$	$-\frac{2}{h_+ h_-}$	$\frac{2}{(h_+ + h_-)h_+}$

Letting $v = \sigma^2/2$ and applying the differencing schemes to the backward PDE (158), we get its (spatial) finite difference approximation, written in matrix-vector form, as follows

$$\frac{\partial U}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} - \mu\frac{\partial U}{\partial x} + rU \Rightarrow \frac{\partial U}{\partial t} = MU, \qquad M = rI - D_v \Delta_{xx} - D_\mu \Delta_x \tag{169}$$

where the D_v is an $n \times n$ diagonal matrix converted from the $n \times 1$ vector v and I denotes an identity matrix. It shows that we can construct a tridiagonal matrix M, which may be a function of t and x, to perform the spatial finite difference approximation for the PDE. It should be emphasized that (169) is only valid for interior grid points. For those grid points at boundaries, further treatment must be taken into account for boundary conditions. For now, let us assume that this approximation is applicable for all the grid points. In a similar manner, we can write the finite difference approximation for the forward PDE (156) as

$$\frac{\partial p}{\partial t}=\frac{1}{2}\frac{\partial^2(\sigma^2 p)}{\partial x^2}-\frac{\partial(\mu p)}{\partial x}\Longrightarrow\frac{\partial p}{\partial t}=Fp,\qquad F=\Delta_{xx}D_v-\Delta_x D_\mu \tag{170}$$

where F is again a tridiagonal matrix that can be dependent on t and x. Since the partial derivatives are taken on products of two functions, both of which can be function of x, the operators (e.g., Δ_{xx} and D_v) do not commute. Unlike what we see in (169), we must flip the matrix multiplication in (170), e.g., from $D_v\Delta_{xx}$ to $\Delta_{xx}D_v$, to reflect the relation.

6.2.3. Temporal Discretization

For the temporal discretization, let us use the backward equation as an example. As mentioned, the 1D PDE in (169) evolves backwards in time. The discretization in time defines the following temporal finite difference approximation

$$\frac{U_t-U_{t-\delta}}{\delta}=\theta M_{t-\delta}U_{t-\delta}+(1-\theta)M_tU_t\Longrightarrow(I+\theta\delta M_{t-\delta})U_{t-\delta}=(I-(1-\theta)\delta M_t)U_t \tag{171}$$

At time t, the U_t is known, we evolve it backwards to $U_{t-\delta}$ for one time step from t to $t-\delta$. The scheme becomes explicit when $\theta=0$, implicit when $\theta=1$ and Crank-Nicolson when $\theta=0.5$. In practice, Crank-Nicolson scheme is often in favor due to its second order accuracy in time. However, it is also well known that the Crank-Nicolson scheme may exhibit localized oscillations for discontinuous terminal conditions if the time step is too coarse relative to the spatial step. A remedy proposed by Rannacher is to take two fully implicit time steps ($\theta=1$) before switching to Crank-Nicolson ($\theta=0.5$) time-stepping. This solution is also known as Rannacher time-stepping [16].

6.2.4. Boundary Conditions

As presented in Table 6.1, the PDE boundary conditions must be specified. In order to handle the boundary condition properly, we consider a vector $\widetilde{U}$ extended (fictitiously) from the vector U with two more *ghost points* added, g_- and g_+, shown as below

$$\underset{(n+2)\times 1}{\widetilde{U}} = \begin{bmatrix} g_- \\ u_0 \\ u_1 \\ \vdots \\ u_{m-1} \\ u_m \\ g_+ \end{bmatrix} \tag{172}$$

where the g_- and g_+ are devised for a specific boundary condition. With the help of the ghost points, all the grid points of U can now be treated as interior points. We can then rewrite the difference equation (171) into

$$\tilde{L}\widetilde{U}_{t-\delta} = \tilde{R}\widetilde{U}_t, \qquad \underset{n\times(n+2)}{\tilde{L}} = \tilde{I} + \theta\delta\widetilde{M}_{t-\delta}, \qquad \underset{n\times(n+2)}{\tilde{R}} = \tilde{I} - (1-\theta)\delta\widetilde{M}_t \tag{173}$$

Both $\tilde{L}$ and $\tilde{R}$ are $n \times (n+2)$ matrices with 3 diagonal entries

$$\underset{n\times(n+2)}{\tilde{L}} = \begin{bmatrix} l_0^- & l_0 & l_0^+ & & & & \\ & l_1^- & l_1 & l_1^+ & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & l_{m-1}^- & l_{m-1} & l_{m-1}^+ & \\ & & & & l_m^- & l_m & l_m^+ \end{bmatrix}, \qquad \underset{n\times(n+2)}{\tilde{R}} = \begin{bmatrix} r_0^- & r_0 & r_0^+ & & & & \\ & r_1^- & r_1 & r_1^+ & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & r_{m-1}^- & r_{m-1} & r_{m-1}^+ & \\ & & & & r_m^- & r_m & r_m^+ \end{bmatrix} \tag{174}$$

and the extended identity matrix defined as

$$\underset{n\times(n+2)}{\tilde{I}} = \begin{bmatrix} 0 & 1 & 0 & & & & \\ & 0 & 1 & 0 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & 0 & 1 & 0 & \\ & & & & 0 & 1 & 0 \end{bmatrix} \tag{175}$$

The lumped matrix $\widetilde{M}$ is an $n \times (n+2)$ matrix, similar to the M in (169), with the extra leading and tailing column added to take advantage of the ghost points for boundary conditions. With the help of the ghost points, the first and the last row of $\widetilde{M}$ can be constructed in the same manner as for the interior grid points. Further notice that, because for every time step we know U_t and want to solve for $U_{t-\delta}$, the (173) can be split into two steps with an intermediate $n \times 1$ vector V, i.e., an explicit step $V = \tilde{R}\tilde{U}_t$ followed by an implicit step $\tilde{L}\tilde{U}_{t-\delta} = V$.

For the explicit step, it is merely a matrix-vector multiplication to form V. However, in each time step, we generally have U_t, which has no explicit ghost points. We must transform the multiplication equivalently to

$$V = \tilde{R}\tilde{U}_t = RU_t + S \tag{176}$$

with an $n \times n$ square matrix R and an auxiliary vector S. The matrix R is basically the same as the matrix $\tilde{R}$ with the first and the last column of $\tilde{R}$ removed, and then replacing the first and the last row of the resulted matrix with 4 new parameters: q_0, q_0^+, q_m^- and q_m. The $n \times 1$ vector S has zeros everywhere except for its first and last entry s_0 and s_m (i.e., nonzero entries for the boundary nodes). This is shown below

$$\underbrace{\begin{bmatrix} v_0 \\ \vdots \\ v_i \\ \vdots \\ v_m \end{bmatrix}}_{\substack{V\\n\times 1}} = \underbrace{\begin{bmatrix} q_0 & q_0^+ & & & \\ r_1^- & r_1 & r_1^+ & & \\ & \cdot & \cdot & \cdot & \\ & & r_{m-1}^- & r_{m-1} & r_{m-1}^+ \\ & & & q_m^- & q_m \end{bmatrix}}_{\substack{R\\n\times n}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}_t}_{\substack{U_t\\n\times 1}} + \underbrace{\begin{bmatrix} s_0 \\ 0 \\ \vdots \\ 0 \\ s_m \end{bmatrix}}_{\substack{S\\n\times 1}} \tag{177}$$

By (176), we must have the following equations

$$\begin{aligned} v_0 &= q_0 u_0 + q_0^+ u_1 + s_0 = r_0^- g_- + r_0 u_0 + r_0^+ u_1 \\ v_m &= q_m^- u_{m-1} + q_m u_m + s_m = r_m^- u_{m-1} + r_m u_m + r_m^+ g_+ \end{aligned} \tag{178}$$

Providing a boundary condition, we are able to find g_- and g_+, then determine R and S by (178), and finally derive V from existing U_t by (176).

For the implicit step, we need to inverse a matrix for the solution $U_{t-\delta}$. Instead of working with $\tilde{L}\tilde{U}_{t-\delta} = V$ directly, we again transform the equation to an equivalent form

$$LU_{t-\delta} + Z = \tilde{L}\tilde{U}_{t-\delta} = V \tag{179}$$

with an $n \times n$ square matrix L and an auxiliary vector Z. The matrix L is constructed similarly as what we have done for matrix R. It is basically the same as the matrix $\tilde{L}$ by removing the first and the last column of $\tilde{L}$, and then replacing the first and the last row of the resulted square matrix with 4 parameters: p_0, p_0^+, p_m^- and p_m. The $n \times 1$ vector Z, which is also similar to the S, has zeros everywhere except for its first and last entry z_0 and z_m. This is shown as follows

$$\underbrace{\begin{bmatrix} p_0 & p_0^+ & & & \\ l_1^- & l_1 & l_1^+ & & \\ & \cdot & \cdot & \cdot & \\ & & l_{m-1}^- & l_{m-1} & l_{m-1}^+ \\ & & & p_m^- & p_m \end{bmatrix}}_{\substack{L \\ n\times n}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}_{t-\delta}}_{\substack{U_{t-\delta} \\ n\times 1}} + \underbrace{\begin{bmatrix} z_0 \\ 0 \\ \vdots \\ 0 \\ z_m \end{bmatrix}}_{\substack{Z \\ n\times 1}} = \underbrace{\begin{bmatrix} v_0 \\ \vdots \\ v_i \\ \vdots \\ v_m \end{bmatrix}}_{\substack{V \\ n\times 1}} \tag{180}$$

By (179), we can derive the following equations

$$\begin{aligned} p_0 u_0 + p_0^+ u_1 + z_0 &= l_0^- g_- + l_0 u_0 + l_0^+ u_1 \\ p_m^- u_{m-1} + p_m u_m + z_m &= l_m^- u_{m-1} + l_m u_m + l_m^+ g_+ \end{aligned} \tag{181}$$

Providing a boundary condition, we are able to find g_- and g_+, then determine L and Z by (181), and finally solve for $U_{t-\delta}$ from the known V by (179) .

Notice that there is a perfect symmetry in (178) and (181), and hence we will only derive the expressions for parameters in L and Z. The conclusions can be easily adapted for the R and S by the symmetry. In the following sub-sections, we will discuss how these parameters can be determined under different boundary conditions.

6.2.4.1. *Neumann Boundary Condition*

Neumann boundary condition assumes the first order derivative is known at the boundary, that is $\partial U/\partial x = f$ at end points. With the ghost points, we can approximate the first derivative at end points, x_0 and x_m, using (166)

$$\begin{aligned} \frac{\partial U(x_0)}{\partial x} &\approx \frac{h_-}{h_+ + h_-}\frac{u_1 - u_0}{h_+} + \frac{h_+}{h_+ + h_-}\frac{u_0 - g_-}{h_-} = f_0 \\ \frac{\partial U(x_m)}{\partial x} &\approx \frac{h_-}{h_+ + h_-}\frac{g_+ - u_m}{h_+} + \frac{h_+}{h_+ + h_-}\frac{u_m - u_{m-1}}{h_-} = f_m \end{aligned} \tag{182}$$

Further assuming equal spacing around the end points, e.g., $h_+ = h_- = h_0 = x_1 - x_0$ for the boundary at x_0 and $h_+ = h_- = h_m = x_m - x_{m-1}$ for boundary at x_m, we can derive

$$g_- = u_1 - 2h_0 f_0, \qquad g_+ = u_{m-1} + 2h_m f_m \tag{183}$$

From (181), we have

$$\begin{aligned} &p_0 = l_0, \qquad p_0^+ = l_0^+ + l_0^-, \qquad z_0 = -2l_0^- h_0 f_0 \\ &p_m^- = l_m^- + l_m^+, \qquad p_m = l_m, \qquad z_m = 2l_m^+ h_m f_m \end{aligned} \tag{184}$$

6.2.4.2. *Convexity Boundary Condition*

Convexity boundary condition assumes the second order derivative is known at the boundary, that is $\partial^2 U/\partial x^2 = f$ at end points. With the ghost points, we can approximate the second derivative at end points, x_0 and x_m, using (166)

$$\begin{aligned} \frac{\partial^2 U(x_0)}{\partial x^2} &\approx \frac{2}{h_+ + h_-}\left(\frac{u_1 - u_0}{h_+} - \frac{u_0 - g_-}{h_-}\right) = f_0 \\ \frac{\partial^2 U(x_m)}{\partial x^2} &\approx \frac{2}{h_+ + h_-}\left(\frac{g_+ - u_m}{h_+} - \frac{u_m - u_{m-1}}{h_-}\right) = f_m \end{aligned} \tag{185}$$

Further assuming equal spacing around the end points, e.g., $h_+ = h_- = h_0 = x_1 - x_0$ for the boundary at x_0 and $h_+ = h_- = h_m = x_m - x_{m-1}$ for boundary at x_m, we can derive

$$\begin{aligned} g_- &= 2u_0 - u_1 + h_0^2 f_0 \\ g_+ &= -u_{m-1} + 2u_m + h_m^2 f_m \end{aligned} \tag{186}$$

From (181), we have

$$\begin{aligned} &p_0 = l_0 + 2l_0^-, \qquad p_0^+ = l_0^+ - l_0^-, \qquad z_0 = l_0^- h_0^2 f_0 \\ &p_m^- = l_m^- - l_m^+, \qquad p_m = l_m + 2l_m^+, \qquad z_m = l_m^+ h_m^2 f_m \end{aligned} \tag{187}$$

6.2.4.3. *Zero Gamma Boundary Condition*

Zero gamma boundary condition assumes the second order derivative equals the first order derivative at the boundary, that is $\partial^2 U/\partial x^2 = \partial U/\partial x$ at end points. With the ghost points, we can approximate the relationship at x_0 and x_m, using (166)

$$\frac{\partial^2 U(x_0)}{\partial x^2} = \frac{\partial U(x_0)}{\partial x} \Longrightarrow \frac{h_- - 2}{h_+}(u_1 - u_0) + \frac{h_+ + 2}{h_-}(u_0 - g_-) = 0 \tag{188}$$

$$\frac{\partial^2 U(x_m)}{\partial x^2} = \frac{\partial U(x_m)}{\partial x} \Longrightarrow \frac{h_- - 2}{h_+}(g_+ - u_m) + \frac{h_+ + 2}{h_-}(u_m - u_{m-1}) = 0$$

Further assuming equal spacing around the end points, e.g., $h_+ = h_- = h_0 = x_1 - x_0$ for the boundary at x_0 and $h_+ = h_- = h_m = x_m - x_{m-1}$ for the boundary at x_m, we can derive

$$\begin{aligned} g_- &= \frac{4}{h_0 + 2}u_0 + \frac{h_0 - 2}{h_0 + 2}u_1 \\ g_+ &= \frac{h_m + 2}{h_m - 2}u_{m-1} - \frac{4}{h_m - 2}u_m \end{aligned} \tag{189}$$

From (181), we have

$$\begin{aligned} &p_0 = l_0 + \frac{4}{h_0 + 2}l_0^-, \qquad p_0^+ = l_0^+ + \frac{h_0 - 2}{h_0 + 2}l_0^-, \qquad z_0 = 0 \\ &p_m^- = l_m^- + \frac{h_m + 2}{h_m - 2}l_m^+, \qquad p_m = l_m - \frac{4}{h_m - 2}l_m^+, \qquad z_m = 0 \end{aligned} \tag{190}$$

6.2.4.4. *Dirichlet Boundary Condition*

Dirichlet boundary condition assumes the function value is known at the boundaries, that is $u_0 = f_0$ and $u_m = f_m$. Since this boundary condition has nothing to do with the ghost points, the treatment is different from the other boundary conditions. Let us first consider the implicit step (179) where we solve for $U_{t-\delta}$ from existing V. Since the u_0 and u_m are always known from Dirichlet boundary conditions, we may construct the linear system as

$$\underbrace{\begin{bmatrix} 1 & 0 & & & \\ l_1^- & l_1 & l_1^+ & & \\ & \cdot & \cdot & \cdot & \\ & & l_{m-1}^- & l_{m-1} & l_{m-1}^+ \\ & & & 0 & 1 \end{bmatrix}}_{\substack{L \\ n\times n}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}_{t-\delta}}_{\substack{U_{t-\delta} \\ n\times 1}} + \underbrace{\begin{bmatrix} z_0 \\ 0 \\ \vdots \\ 0 \\ z_m \end{bmatrix}}_{\substack{Z \\ n\times 1}} = \underbrace{\begin{bmatrix} v_0 \\ \vdots \\ v_i \\ \vdots \\ v_m \end{bmatrix}}_{\substack{V \\ n\times 1}} \tag{191}$$

which gives the parameters

$$\begin{aligned} &p_0 = 1, \qquad p_0^+ = 0, \qquad z_0 = v_0 - f_0 \\ &p_m^- = 0, \qquad p_m = 1, \qquad z_m = v_m - f_m \end{aligned} \tag{192}$$

This shows that regardless of the values of v_0 and v_m, and we always have $f_0 = v_0 - z_0$ and $f_m = v_m - z_m$ in the right hand side vector $V - Z$ to ensure $u_0 = f_0$ and $u_m = f_m$. It is only the interior entries of V that matter. This is equivalent to first knowing $u_0 = f_0$ and $u_m = f_m$ and then solving for interior entries u_i for $0 < i < m$ based on these values.

For the explicit step (176) where we find vector V from existing U_t, the parameters can be derived in the same manner

$$\underbrace{\begin{bmatrix} v_0 \\ \vdots \\ v_i \\ \vdots \\ v_m \end{bmatrix}}_{\substack{V \\ n\times 1}} = \underbrace{\begin{bmatrix} q_0 & q_0^+ & & & \\ r_1^- & r_1 & r_1^+ & & \\ & \cdot & \cdot & \cdot & \\ & & r_{m-1}^- & r_{m-1} & r_{m-1}^+ \\ & & & q_m^- & q_m \end{bmatrix}}_{\substack{R \\ n\times n}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_i \\ \vdots \\ u_m \end{bmatrix}_t}_{\substack{U_t \\ n\times 1}} + \underbrace{\begin{bmatrix} s_0 \\ 0 \\ \vdots \\ 0 \\ s_m \end{bmatrix}}_{\substack{S \\ n\times 1}} \tag{193}$$

which gives the parameters

$$\begin{aligned} &q_0 = 1, \qquad q_0^+ = 0, \qquad s_0 = f_0 - u_0 \\ &q_m^- = 0, \qquad q_m = 1, \qquad s_m = f_m - u_m \end{aligned} \tag{194}$$

This is equivalent to first calculating $V = \tilde{R}\tilde{U}_t$ for interior entries ($0 < i < m$) and then setting $v_0 = f_0$ and $v_m = f_m$. The interior entries v_i for $0 < i < m$ have no direct dependency on the v_0 and v_m.

The Table 6.3 (and Table 6.4) below summarizes the changes to the first and last row of matrix R and vector S in explicit step (and matrix L and vector Z in implicit step) for the associated boundary conditions. Note that the treatment is essentially the same in the two steps due to symmetry, except for the vector S and Z in Dirichlet boundary condition.

Table 6.3: Matrix R and vector S by boundary conditions for *explicit* step $V = RU + S$

Type	$R_{0,0} = q_0$	$R_{0,1} = q_0^+$	s_0	$R_{m,m-1} = q_m^-$	$R_{m,m} = q_m$	s_m
Dirichlet	1	0	$f_0 - u_0$	0	1	$f_m - u_m$
Neumann	r_0	$r_0^+ + r_0^-$	$-2r_0^- h_0 f_0$	$r_m^- + r_m^+,$	r_m	$2r_m^+ h_m f_m$
Convexity	$r_0 + 2r_0^-$	$r_0^+ - r_0^-$	$r_0^- h_0^2 f_0$	$r_m^- - r_m^+$	$r_m + 2r_m^+$	$r_m^+ h_m^2 f_m$
ZeroGamma	$r_0 + \frac{4}{h_0 + 2} r_0^-$	$r_0^+ + \frac{h_0 - 2}{h_0 + 2} r_0^-$	0	$r_m^- + \frac{h_m + 2}{h_m - 2} r_m^+$	$r_m - \frac{4}{h_m - 2} r_m^+$	0

Table 6.4 Matrix L and vector Z by boundary conditions for *implicit* step $U = L^{-1}(V - Z)$

Type	$L_{0,0} = p_0$	$L_{0,1} = p_0^+$	z_0	$L_{m,m-1} = p_m^-$	$L_{m,m} = p_m$	z_m
Dirichlet	1	0	$v_0 - f_0$	0	1	$v_m - f_m$
Neumann	l_0	$l_0^+ + l_0^-$	$-2l_0^- h_0 f_0$	$l_m^- + l_m^+,$	l_m	$2l_m^+ h_m f_m$
Convexity	$l_0 + 2l_0^-$	$l_0^+ - l_0^-$	$l_0^- h_0^2 f_0$	$l_m^- - l_m^+$	$l_m + 2l_m^+$	$l_m^+ h_m^2 f_m$
ZeroGamma	$l_0 + \dfrac{4}{h_0 + 2} l_0^-$	$l_0^+ + \dfrac{h_0 - 2}{h_0 + 2} l_0^-$	0	$l_m^- + \dfrac{h_m + 2}{h_m - 2} l_m^+$	$l_m - \dfrac{4}{h_m - 2} l_m^+$	0

The boundary conditions for forward equation can be treated in a similar manner. However, as one can see from (170), the coefficient functions μ and v (for $v = \sigma^2/2$) are involved within the partial derivatives. We need to impose some boundary conditions for these functions in order to derive their corresponding ghost point values, which are required to form extended diagonal matrices, e.g., D_μ and D_v. These matrices will be subsequently used to assemble the matrix $\Delta_{xx} D_v$ and $\Delta_x D_\mu$ and eventually the matrix F for numerical solution of the PDE. An intuitive boundary condition can be used for this purpose, such as zero convexity boundary condition, i.e., we assume linear extrapolation at boundaries for these functions.

6.3. Finite Difference Solver in Multi-Dimension

Suppose we have an m-dimensional initial value equation with proper initial and boundary conditions of the form

$$\frac{\partial U}{\partial t} = \sum_{i=1}^{m} a_i \frac{\partial U}{\partial x_i} + \sum_{i,j=1}^{m} b_{ij} \frac{\partial^2 U}{\partial x_i \partial x_j} + cU \tag{195}$$

where coefficients a_i, b_{ij} and c can be functions of t and x_i's. Such PDE's are commonly classified as convection-diffusion equations. In the context of derivative pricing, they are usually parabolic (as the partial derivative with respect to t is only up to the first order). As the dimensionality of the space increases, the complexity of finite difference schemes and the difficulty to implement such algorithms increase remarkably. The system that combines the operators from all dimensions is no longer tridiagonal and may possesses a large bandwidth. Though still very sparse, it may demand a suitable sparse matrix technique and can be computationally intensive.

To resolve this "curse of dimensionality", researchers developed alternating direction implicit (ADI) methods and locally one dimensional (LOD) methods to solve multi-dimensional parabolic equations on a square domain using finite differences. The main idea of the two types of methods is to split the simultaneous application of operators from all spatial dimensions into sequential operator applications, one for each dimension, so that the 2D or 3D problem can be solved as two or three consecutive 1D problems, where tridiagonal matrices can still apply. A brief introduction to these methods can be found in [17] [18] [19] [20]. The intuition behind this is straightforward. When a PDE contains different terms expressing different physics, it is natural to use different numerical methods for different physical processes. This can optimize and simplify the overall solution process.

To sketch out these methods, we may write an m-dimensional initial value equation as

$$\frac{\partial U}{\partial t} = \mathcal{L}U, \qquad \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \cdots + \mathcal{L}_m \tag{196}$$

where $\mathcal{L}$ is some operator (for example, $\mathcal{L}_1 = a_1 \frac{\partial}{\partial x_1} + b_{11} \frac{\partial^2}{\partial x_1^2}$, and so on). While $\mathcal{L}$ is not necessarily linear, suppose that it can be at least written as a linear sum of m pieces, one for each dimension, which act additively on U. Assume that for each of the pieces, we already know a differencing scheme for updating U from timestep n to timestep $n+1$ with $\delta = t_{n+1} - t_n$, valid if that piece of the operator were the only one on the right-hand side. The ADI methods evolve the state variable U in each timestep in multiple stages, treating only one dimension implicitly (and the other dimensions explicitly) at each stage. The following updating steps show a typical ADI scheme

$$\begin{aligned} \frac{U_{n+1/m} - U_n}{\delta/m} &= \mathcal{L}_1 U_{n+1/m} + \mathcal{L}_2 U_n + \cdots + \mathcal{L}_m U_n \\ \frac{U_{n+2/m} - U_{n+1/m}}{\delta/m} &= \mathcal{L}_1 U_{n+1/m} + \mathcal{L}_2 U_{n+2/m} + \cdots + \mathcal{L}_m U_{n+1/m} \end{aligned} \tag{197}$$

$$\cdots$$

$$\frac{U_{n+1} - U_{n+(m-1)/m}}{\delta/m} = \mathcal{L}_1 U_{n+(m-1)/m} + \mathcal{L}_2 U_{n+(m-1)/m} + \cdots + \mathcal{L}_m U_{n+1}$$

The LOD methods, on the other hand, divide each timestep into fractional timesteps with simpler operators, such that each fractional step contains only the derivatives of the variable of interest

$$\begin{aligned}
&\frac{U_{n+1/m} - U_n}{\delta} = \mathcal{L}_1 U_{n+1/m} \\
&\frac{U_{n+2/m} - U_{n+1/m}}{\delta} = \mathcal{L}_2 U_{n+2/m} \\
&\cdots \\
&\frac{U_{n+1} - U_{n+(m-1)/m}}{\delta} = \mathcal{L}_m U_{n+1}
\end{aligned} \tag{198}$$

It is worth noting that in ADI schemes like (197), the intermediate quantities $U_{n+i/m}$ for $i = 1, \cdots, m-1$ are approximation of U_{n+1}, numerical boundary conditions can be directly derived from those of the original problem. However, in LOD schemes, like (198), the intermediate steps are non-physical and are purely mathematical construct. The accuracy of LOD schemes depends on the approximation of boundary conditions at the fractional timesteps, which in many cases is not trivial to implement. The simplest approximation method is to use the exact boundary conditions given by the full PDE. This is not an issue for scheme like (198), which is just first order accurate in time. However, other higher order LOD schemes using this approximation may suffer from accuracy order reduction [21] [22] [23].

In our applications, we use scheme (198), which is also known as *Marchuk-Yanenko* operator splitting [24] [25] [26], to solve multi-dimensional parabolic PDE. Unfortunately, the mixed derivative prevents this from occurring. So, it is handled explicitly. We are in favor of this method because

- It is a fully implicit scheme (except for the cross term) that ensures unconditional stability. In contrast, in the ADI scheme (197), each stage we solve for one dimension implicitly, relying on explicit derivatives of the other dimensions. If there were abrupt changes due to, say, non-smooth payoffs, the explicit derivative might lead to difficulty in solving the problem.

- It is computationally efficient, easy to implement. Although it is just first order accurate in time, it suffices for our applications. Schemes with higher order accuracy generally require implementation of complex approximation of boundary conditions for intermediate steps and special treatment of mixed derivatives to avoid accuracy order reduction.

In the following, we discuss in details about implementation of this method in solving backward and forward PDEs arising from derivative pricing applications. The example is in 2D.

6.3.1. Backward PDE

Suppose we have a 2D backward PDE in a general form with proper initial and boundary conditions

$$\frac{\partial U}{\partial t} = a\frac{\partial U}{\partial x} + \alpha\frac{\partial U}{\partial y} + b\frac{\partial^2 U}{\partial x^2} + \beta\frac{\partial^2 U}{\partial y^2} + c\frac{\partial^2 U}{\partial x \partial y} + fU \tag{199}$$

where coefficients $a, b, c, f, \alpha, \beta$ can be functions of t, x, y. To solve the equation, we split the full PDE (195) into two 1D PDE's, each containing only one of the spatial variables, while the mixed derivative is handled explicitly in x PDE

$$\frac{\partial U}{\partial t} = a\frac{\partial U}{\partial x} + b\frac{\partial^2 U}{\partial x^2} + c\frac{\partial^2 U}{\partial x \partial y} + fU, \qquad \frac{\partial U}{\partial t} = \alpha\frac{\partial U}{\partial y} + \beta\frac{\partial^2 U}{\partial y^2} \tag{200}$$

For each time step, from t to $t-1$, we take the result of the previous time step and use it as the start point for the x step. We take the result of the x step and use it as the starting point for the y step as we evolve again over the same time interval. After both steps, we have evolved all two variables for the time step t to $t-1$.

Numerically, we represent coefficients $a, b, c, f, \alpha, \beta$ and function U as $n \times m$ matrices, with n rows representing x direction and m columns representing y direction. Suppose we want to evolve the PDE backwards for a time step from t to $t-1$ with a time interval δ. We first prepare the cross term and zero order term using the known U_t. This is equivalent to compute

$$c \cdot \left(\Delta_x U_t \Delta'_y\right) + f \cdot U_t \tag{201}$$

where the dot operator denotes element-wise multiplication of matrices and prime symbol stands for matrix transpose operation. Both differential operators $\mathit{\Delta}_x$ and $\mathit{\Delta}_y$ are constructed according to Table 6.2. Once the term (201) is obtained, we move to the x step where we solve for $U_{t-\delta/2}$ column by column using fully implicit scheme. This is equivalent to writing

$$\begin{aligned}\frac{U_t - U_{t-\delta/2}}{\delta} &= (D_a \mathit{\Delta}_x + D_b \mathit{\Delta}_{xx}) U_{t-\delta/2} + c \cdot \mathit{\Delta}_x U_t \mathit{\Delta}'_y + f \cdot U_t \Longrightarrow \\ (I + \delta D_a \mathit{\Delta}_x + \delta D_b \mathit{\Delta}_{xx}) U_{t-\delta/2} &= U_t - \delta c \cdot \mathit{\Delta}_x U_t \mathit{\Delta}'_y - \delta f \cdot U_t\end{aligned} \tag{202}$$

where the diagonal matrices D_a and D_b are constructed from corresponding columns of a and b respectively. Note that a and b can be functions of both x and y, the D_a and D_b may vary when we iterate through columns. Once the $U_{t-\delta/2}$ is computed, we carry on with the y step where we solve for every row of U_{t-1} in the same manner, such that

$$\frac{U'_{t-\delta/2} - U'_{t-1}}{\delta} = \left(D_\alpha \mathit{\Delta}_y + D_\beta \mathit{\Delta}_{yy}\right) U'_{t-1} \Longrightarrow \left(I + \delta D_\alpha \mathit{\Delta}_y + \delta D_\beta \mathit{\Delta}_{yy}\right) U'_{t-1} = U'_{t-\delta/2} \tag{203}$$

where the D_α and D_β are derived from corresponding rows of α and β respectively. It is worth mentioning that in the steps (201), (202) and (203), the matrices and vectors must be adjusted consistently to the boundary conditions, as shown in Table 6.4. This has been discussed in details in Section 6.2.4. As an approximation, exact boundary conditions given by the full PDE are used for the intermediate steps.

6.3.2. Forward PDE

In the case of a forward PDE, it often admits the following general form

$$\frac{\partial p}{\partial t} = \frac{\partial (ap)}{\partial x} + \frac{\partial (\alpha p)}{\partial y} + \frac{\partial^2 (bp)}{\partial x^2} + \frac{\partial^2 (\beta p)}{\partial y^2} + \frac{\partial^2 (cp)}{\partial x \partial y} + fp \tag{204}$$

where p can be the transition probability density. The treatment is nothing special. We still break the full PDE down into the following two PDE's

$$\frac{\partial p}{\partial t} = \frac{\partial (ap)}{\partial x} + \frac{\partial^2 (bp)}{\partial x^2} + \frac{\partial^2 (cp)}{\partial x \partial y} + fp, \qquad \frac{\partial p}{\partial t} = \frac{\partial (\alpha p)}{\partial y} + \frac{\partial^2 (\beta p)}{\partial y^2} \tag{205}$$

and then solve implicitly for the x step followed by the y step. The only difference from the backward PDE is that the partial derivatives are now taken on the products of two functions instead of function p alone. To take care of this, we will follow the same strategy as stated in Section 6.2.4, which assumes zero convexity boundary condition for these coefficient functions and then finds their corresponding ghost point values to form the extended diagonal matrices for matrix assembling. To illustrate the steps, let us define coefficients $a, b, c, f, \alpha, \beta$ and function p as $n \times m$ matrices, with n rows representing x direction and m columns representing y direction. For the cross term, we estimate it explicitly as

$$\mathit{\Delta}_x(c \cdot p_t)\mathit{\Delta}'_y + f \cdot p_t \tag{206}$$

And for the x step, we implicitly solve for every column of $p_{t+\delta/2}$ by

$$\begin{aligned} &\frac{p_{t+\delta/2} - p_t}{\delta} = (\mathit{\Delta}_x D_a + \mathit{\Delta}_{xx} D_b) p_{t+\delta/2} + \mathit{\Delta}_x(c \cdot p_t)\mathit{\Delta}'_y + f \cdot p_t \Longrightarrow \\ &(I - \delta \mathit{\Delta}_x D_a - \delta \mathit{\Delta}_{xx} D_b) p_{t+\delta/2} = p_t + \delta \mathit{\Delta}_x(c \cdot p_t)\mathit{\Delta}'_y + \delta f \cdot p_t \end{aligned} \tag{207}$$

Once the $p_{t+\delta/2}$ is ready, we then solve the y step implicitly for every row of p_{t+1} by

$$\frac{p'_{t+1} - p'_{t+\delta/2}}{\delta} = \left(\mathit{\Delta}_y D_\alpha + \mathit{\Delta}_{yy} D_\beta\right) p'_{t+1} \Longrightarrow \left(I - \delta \mathit{\Delta}_y D_\alpha - \delta \mathit{\Delta}_{yy} D_\beta\right) p'_{t+1} = p'_{t+\delta/2} \tag{208}$$

Note that different from backward equation, we have switched the sequence of matrix multiplication, e.g., $\mathit{\Delta}_x D_a$ in forward equation versus $D_a \mathit{\Delta}_x$ in backward equation, to account for the differences in partial derivatives. Again, the matrices and vectors must also be adjusted to take boundary conditions into account. The exact boundary conditions are again used as an approximation for the intermediate steps.

7. THE HEATH-JARROW-MORTON FRAMEWORK

In this chapter we will discuss *Heath-Jarrow-Morton* (HJM) Framework, a general no-arbitrage framework for interest rate models. Since the description of the single-factor HJM model is readily available [27], without loss of generality, our focus will be on the multi-factor version of the HJM model, which can be easily reduced to single-factor model.

7.1. Forward Rate

The HJM Framework assumes, for a *fixed* maturity T, the instantaneous forward rate $f_{t,T}$ evolves, under a certain probability measure (e.g., physical measure $\mathbb{P}$), as a diffusion process defined by

$$df_{t,T} = \alpha_{t,T}dt + \mathbb{1}'\beta_{t,T}dW_t \tag{209}$$

where we use a prime symbol (e.g., $\mathbb{1}'$) to denote a matrix transpose operation and define the $n \times 1$ vector-valued and $n \times n$ matrix-valued functions as

$$\underset{n\times 1}{\mathbb{1}} = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}, \qquad \underset{n\times n}{\beta_{t,T}} = \mathrm{Diag}\begin{bmatrix} \vdots \\ \beta_{i;t,T} \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{dW_t} = \begin{bmatrix} \vdots \\ dW_{i;t} \\ \vdots \end{bmatrix}, \qquad dW_t dW_t' = \underset{n\times n}{\rho}\, dt \tag{210}$$

The $\mathbb{1}$ denotes an all-ones vector used to aggregate vector/matrix elements. The $\beta_{t,T}$ is a diagonal matrix denotes an adapted volatility process. The $dW_t \in \mathbb{R}^{n\times 1}$ denotes a column vector of Brownian motions under physical measure $\mathbb{P}$, whose correlation matrix ρ is given by $dW_t dW_t' = \rho dt$. The advantage of modeling forward rate is that the current term structure of the forward rate is, by construction, an input of the model.

The derivation starts with a zero-coupon bond, a market tradable asset, whose value is given by the instantaneous forward rates through (40)

$$P_{t,T} = \exp\left(-\int_t^T f_{t,u}du\right) \tag{211}$$

The bond price dynamics is a total differential with respect to t, which can be calculated via Ito's lemma

$$\frac{dP_{t,T}}{P_{t,T}} = d\left(-\int_t^T f_{t,u}du\right) + \frac{1}{2}d\left(-\int_t^T f_{t,u}du\right)d\left(-\int_t^T f_{t,u}du\right) \qquad \text{and} \tag{212}$$

$$d\left(-\int_t^T f_{t,u}du\right) = f_{t,t}dt - \int_t^T df_{t,u}du = r_t dt - \int_t^T (\alpha_{t,u}dt)du - \mathbb{1}'\int_t^T (\beta_{t,u}dW_t)du$$

Since integrals can be regarded as a limit of Riemann sums, we can reverse the order of the integration in (212) such that

$$\int_t^T (\alpha_{t,u}dt)du = \left(\int_t^T \alpha_{t,u}du\right)dt \quad \text{and} \quad \int_t^T (\beta_{t,u}dW_t)du = \left(\int_t^T \beta_{t,u}du\right)dW_t \tag{213}$$

If we define

$$a_{t,T} = \int_t^T \alpha_{t,u}du \quad \text{and} \quad \underset{n\times n}{b_{t,T}} = \int_t^T \beta_{t,u}du \tag{214}$$

we shall have

$$d\left(-\int_t^T f_{t,u}du\right) = r_t dt - a_{t,T}dt - \mathbb{1}'b_{t,T}dW_t \tag{215}$$

and hence the dynamics of bond price reads

$$\frac{dP_{t,T}}{P_{t,T}} = \left(r_t - a_{t,T} + \frac{1}{2}\mathbb{1}'b_{t,T}\rho b_{t,T}\mathbb{1}\right)dt - \mathbb{1}'b_{t,T}dW_t \tag{216}$$

The negative sign in front of the diffusion term indicates the movement of bond price is negatively correlated with the forward rates. If we define a market price of risk vector λ_t, such that it satisfies the following equation

$$\mathbb{1}'b_{t,T}\lambda_t = a_{t,T} - \frac{1}{2}\mathbb{1}'b_{t,T}\rho b_{t,T}\mathbb{1} \tag{217}$$

there will be infinite equations, one for each bond with a different maturity T. However, the λ_t must be unique (i.e., independent of T), otherwise arbitrage opportunity arises. Since the bond is a tradable asset, its price must admit a drift at risk-free rate r_t under risk neutral measure $\mathbb{Q}$, that is

$$\frac{dP_{t,T}}{P_{t,T}} = r_t dt - \mathbb{1}'b_{t,T}(dW_t + \lambda_t dt) = r_t dt - \mathbb{1}'b_{t,T}d\widetilde{W}_t \tag{218}$$

where $d\widetilde{W}_t = dW_t + \lambda_t dt$ is an n-dimensional Brownian motion under $\mathbb{Q}$.

The forward rate dynamics under $\mathbb{Q}$ can be derived in a similar manner. We first apply $\frac{\partial}{\partial T}$ to (217), which gives

$$\mathbb{1}'\beta_{t,T}\lambda_t = \alpha_{t,T} - \mathbb{1}'\beta_{t,T}\rho b_{t,T}\mathbb{1} \tag{219}$$

and hence from (209)

$$df_{t,T} = \alpha_{t,T}dt + \mathbb{1}'\beta_{t,T}\big(d\widetilde{W}_t - \lambda_t dt\big) = \mathbb{1}'\beta_{t,T}\rho b_{t,T}\mathbb{1}dt + \mathbb{1}'\beta_{t,T}d\widetilde{W}_t \tag{220}$$

As one can see, under risk neutral measure $\mathbb{Q}$, the drift term of the forward rate dynamics is fully determined by the volatility term. This is known as the *HJM no-arbitrage condition*.

We know from (41) that $f_{t,T}$ is a martingale under the T-forward measure $\mathbb{Q}^T$, where the associated numeraire $P_{t,T}$ has a volatility term $-b_{t,T}$. According to (23), we have

$$dW_t^T = d\widetilde{W}_t + \rho b_{t,T}\mathbb{1}dt \tag{221}$$

as a Brownian motion under $\mathbb{Q}^T$. The dynamics of $f_{t,T}$ becomes driftless after change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$, such that

$$df_{t,T} = \mathbb{1}'\beta_{t,T}\big(d\widetilde{W}_t + \rho b_{t,T}\mathbb{1}dt\big) = \mathbb{1}'\beta_{t,T}dW_t^T \tag{222}$$

In short, the following two equations summarize the above discussion

$$\begin{aligned}\frac{dP_{t,T}}{P_{t,T}} &= r_t dt - \mathbb{1}'b_{t,T}d\widetilde{W}_t = \big(r_t + \mathbb{1}'b_{t,T}\rho b_{t,T}\mathbb{1}\big)dt - \mathbb{1}'b_{t,T}dW_t^T \\ df_{t,T} &= \mathbb{1}'\beta_{t,T}\rho b_{t,T}\mathbb{1}dt + \mathbb{1}'\beta_{t,T}d\widetilde{W}_t = \mathbb{1}'\beta_{t,T}dW_t^T\end{aligned} \tag{223}$$

Given the forward rate dynamics in (223), the instantaneous correlation between two forward rates, $f_{t,T}$ and $f_{t,V}$, can be calculated as

$$\mathrm{Correl}\big(f_{t,T}, f_{t,V}\big) = \frac{\mathbb{1}'\beta_{t,T}\rho\beta_{t,V}\mathbb{1}}{\sqrt{\mathbb{1}'\beta_{t,T}\rho\beta_{t,T}\mathbb{1}}\sqrt{\mathbb{1}'\beta_{t,V}\rho\beta_{t,V}\mathbb{1}}} \tag{224}$$

If the ρ were an identity matrix (i.e., independent Brownian motions), the instantaneous correlation would be no more than a cosine similarity between the two volatility vectors, $\beta_{t,T}\mathbb{1}$ and $\beta_{t,V}\mathbb{1}$. In single-factor models (i.e., dimension = 1), the (224) ends up with a perfect positive correlation for the whole term structure of forward rates regardless of the volatility function chosen. The imposed perfect correlation is obviously unrealistic. The rate term structure dynamics observed in the markets show not only the parallel moves, but also steepening and butterflies. This constraint, however, can be relaxed when the dimensionality of the models is increased. By specifying a proper volatility function, it allows the vectors of the forward rate volatilities to be non-parallel and therefore allows the forward rate correlation to vary over T.

7.2. Short Rate

Integrating $df_{u,T}$ in (223) over u from start time s to present time t gives the forward rate

$$f_{t,T} = f_{s,T} + \mathbb{1}' \int_s^t \beta_{u,T} \rho b_{u,T} du \, \mathbb{1} + \mathbb{1}' \int_s^t \beta_{u,T} d\widetilde{W}_u = f_{s,T} + \mathbb{1}' \int_s^t \beta_{u,T} dW_u^T \tag{225}$$

When $T = t$, we have the short rate (i.e., instantaneous spot rate) under risk neutral measure

$$\begin{gathered} r_t = f_{s,t} + \mathbb{1}' \int_s^t \beta_{u,t} \rho b_{u,t} du \, \mathbb{1} + \int_s^t \mathbb{1}' \beta_{u,t} d\widetilde{W}_u \quad \text{and its dynamics} \\ dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \int_s^t \mathbb{1}' \left(\frac{\partial \beta_{u,t}}{\partial t} \rho b_{u,t} + \beta_{u,t} \rho \beta_{u,t} \right) \mathbb{1} du + \int_s^t \mathbb{1}' \frac{\partial \beta_{u,t}}{\partial t} d\widetilde{W}_u \right) dt + \mathbb{1}' \beta_{t,t} d\widetilde{W}_t \end{gathered} \tag{226}$$

where in the first equation the $f_{s,t}$ reflects the initial forward rate term structure, and the covariance $\int_s^t \beta_{u,t} \rho b_{u,t} du$ is the convexity adjustment comes merely from a measure change from t-forward measure (under which r_t is a martingale) to the risk neutral measure in a continuous manner starting from initial time s. The diffusion term in rate dynamics shows that the instantaneous volatility of the short rate is $\beta_{t,t}$. The first term in drift shows the drift comes partially from the slope of the initial forward rate curve. The second term in drift depends on the history of β as it involves $\beta_{u,t}$ for $u < t$. The third term in drift depends on the history of both β and $d\widetilde{W}$. The second term (if β is stochastic) and the third term are liable to cause the process of r to be non-Markovian, i.e., the drift of r between time t and $t + \Delta t$ may depend not only on the value of r at t, but also on the history of r prior to t [28].

In further analysis, we may reformulate the short rate (226) into an affine function of x_t

$$r_t = f_{s,t} + \mathbb{1}' x_t \tag{227}$$

where we define the $n \times 1$ vector-valued stochastic variable x_t along with its variance y_t (which is an $n \times n$ symmetric-matrix-valued auxiliary variable)

$$\begin{aligned}
& x_t = \int_s^t \beta_{u,t} \rho b_{u,t} du \, \mathbb{1} + \int_s^t \beta_{u,t} d\widetilde{W}_u \quad \text{and its dynamics} \\
& dx_t = \left(\int_s^t \left(\frac{\partial \beta_{u,t}}{\partial t} \rho b_{u,t} + \beta_{u,t} \rho \beta_{u,t} \right) \mathbb{1} du + \int_s^t \frac{\partial \beta_{u,t}}{\partial t} d\widetilde{W}_u \right) dt + \beta_{t,t} d\widetilde{W}_t \\
& y_t = \varphi_{s,t,t,t} = \int_s^t \beta_{u,t} \rho \beta_{u,t} du \quad \text{and its dynamics} \\
& dy_t = \left(\int_s^t \frac{\partial \beta_{u,t}}{\partial t} \rho \beta_{u,t} du + \int_s^t \beta_{u,t} \rho \frac{\partial \beta_{u,t}}{\partial t} du + \beta_{t,t} \rho \beta_{t,t} \right) dt
\end{aligned} \tag{228}$$

To further ease the notation, we define another three $n \times n$ matrix-valued auxiliary variance functions, which will be used repeatedly in the context

$$\begin{aligned}
& \varphi_{s,t,T,V} = \int_s^t \beta_{u,T} \rho \beta_{u,V} du, \qquad [\varphi_{s,t,T,V}]_{i,j} = \int_s^t \beta_{i;u,T} \rho_{ij} \beta_{j;u,V} du \\
& \chi_{s,t,T,V} = \int_s^t \beta_{u,T} \rho b_{u,V} du, \qquad [\chi_{s,t,T,V}]_{i,j} = \int_s^t \beta_{i;u,T} \rho_{ij} b_{j;u,V} du \\
& \psi_{s,t,T,V} = \int_s^t b_{u,T} \rho b_{u,V} du, \qquad [\psi_{s,t,T,V}]_{i,j} = \int_s^t b_{i;u,T} \rho_{ij} b_{j;u,V} du \\
& \xi^2_{s,t,T,V} = \int_s^t (b_{u,V} - b_{u,T}) \rho (b_{u,V} - b_{u,T}) du = \psi_{s,t,V,V} - \psi_{s,t,T,V} - \psi_{s,t,V,T} + \psi_{s,t,T,T}
\end{aligned} \tag{229}$$

In general, the volatility process $\beta_{u,t}$ can be stochastic, hence the distribution of x_t and y_t are not well known. However, under the assumption of deterministic $\beta_{u,t}$, the y_t becomes deterministic and the variable x_t follows a normal distribution characterized by mean $\int_s^t \beta_{u,t} \rho b_{u,t} du \, \mathbb{1}$ and variance y_t.

The non-Markovianess of the short rate r_t can also be identified from the expression of x_t in (228), where the stochastic driver x_t has time t in the stochastic integral not only as the upper bound of integration, but also inside the integrand function. However, if we assume a *separable* form for $\beta_{t,T}$, e.g.

$$\beta_{t,T} = \sigma_t \lambda_T \tag{230}$$

The x_t and y_t become a joint Markovian process, as shown below

$$\begin{aligned} dx_t &= \left(\frac{\partial \log \lambda_t}{\partial t} x_t + y_t \mathbb{1}\right) dt + \lambda_t \sigma_t d\widetilde{W}_t \\ dy_t &= \left(\frac{\partial \log \lambda_t}{\partial t} y_t + y_t \frac{\partial \log \lambda_t}{\partial t} + \lambda_t \sigma_t \rho \sigma_t \lambda_t\right) dt \end{aligned} \tag{231}$$

and so is the short rate r_t. To further extend the model to cover the skew/smile features in rate dynamics, one may assume the $\beta_{t,T}$ to be a stochastic volatility or a local volatility (or both). A summary of such extensions can be found in [29].

7.3. Zero Coupon Bond

The bond price can be expressed as $P_{t,T} = \exp\left(-\int_t^T f_{t,v} dv\right)$ with the forward rate (225)

$$\begin{aligned} P_{t,T} &= \exp\left(-\int_t^T f_{t,v} dv\right) = \frac{P_{s,T}}{P_{s,t}} \exp\left(-\int_t^T \int_s^t \mathbb{1}' \beta_{u,v} \rho b_{u,v} \mathbb{1} du\, dv - \int_t^T \int_s^t \mathbb{1}' \beta_{u,v} d\widetilde{W}_u\, dv\right) \\ &= \frac{P_{s,T}}{P_{s,t}} \exp\left(-\int_s^t \mathbb{1}' \frac{b_{u,T} \rho b_{u,T} - b_{u,t} \rho b_{u,t}}{2} \mathbb{1} du - \int_s^t \mathbb{1}' \left(b_{u,T} - b_{u,t}\right) d\widetilde{W}_u\right) \end{aligned} \tag{232}$$

Alternatively, the bond price can also be derived from $P_{t,T} = \widetilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r_v dv\right)\right]$ with the short rate (226). The same result as in (232) should be reached. We firstly derive the integral of the short rate

$$\begin{aligned} \int_s^t r_v dv &= \int_s^t \left(f_{s,v} + \int_s^v \mathbb{1}' \beta_{u,v} \rho b_{u,v} \mathbb{1} du + \int_s^v \mathbb{1}' \beta_{u,v} d\widetilde{W}_u\right) dv \\ &= -\log P_{s,t} + \int_s^t \int_u^t \mathbb{1}' \beta_{u,v} \rho b_{u,v} \mathbb{1} dv\, du + \int_s^t \int_u^t \mathbb{1}' \beta_{u,v} dv\, d\widetilde{W}_u \\ &= -\log P_{s,t} + \int_s^t \frac{\mathbb{1}' b_{u,t} \rho b_{u,t} \mathbb{1}}{2} du + \int_s^t \mathbb{1}' b_{u,t} d\widetilde{W}_u \end{aligned} \tag{233}$$

As expected, the bond price admits the same expression as in (232)

$$P_{t,T} = \widetilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r_v dv\right)\right] \tag{234}$$

$$= \frac{P_{s,T}}{P_{s,t}} \widetilde{\mathbb{E}}_t \left[\exp\left(\int_s^t \frac{\mathbb{1}' b_{u,t} \rho b_{u,t} \mathbb{1}}{2} du + \int_s^t \mathbb{1}' b_{u,t} d\widetilde{W}_u - \int_s^T \frac{\mathbb{1}' b_{u,T} \rho b_{u,T} \mathbb{1}}{2} du - \int_s^T \mathbb{1}' b_{u,T} d\widetilde{W}_u \right) \right]$$

$$= \frac{P_{s,T}}{P_{s,t}} e^{-\int_s^t \mathbb{1}' \frac{b_{u,T} \rho b_{u,T} - b_{u,t} \rho b_{u,t}}{2} \mathbb{1} du - \int_s^t \mathbb{1}' (b_{u,T} - b_{u,t}) d\widetilde{W}_u} \underbrace{\widetilde{\mathbb{E}}_t \left[e^{-\int_t^T \frac{\mathbb{1}' b_{u,T} \rho b_{u,T} \mathbb{1}}{2} du - \int_t^T \mathbb{1}' b_{u,T} d\widetilde{W}_u} \right]}_{=1}$$

$$= \frac{P_{s,T}}{P_{s,t}} \exp\left(-\int_s^t \mathbb{1}' \frac{b_{u,T} \rho b_{u,T} - b_{u,t} \rho b_{u,t}}{2} \mathbb{1} du - \int_s^t \mathbb{1}' (b_{u,T} - b_{u,t}) d\widetilde{W}_u \right)$$

where we can show that the expectation highlighted above is always equal to 1 even with a stochastic $b_{t,T}$ (with assumption that $b_{t,T}$ is stochastic only in t). The proof is sketched below with an infinitesimal time interval defined as $\delta = \frac{T-t}{n+1}$ for $n \to \infty$

$$\widetilde{\mathbb{E}}_t \left[e^{-\int_t^T \frac{\mathbb{1}' b_{u,T} \rho b_{u,T} \mathbb{1}}{2} du - \int_t^T \mathbb{1}' b_{u,T} d\widetilde{W}_u} \right] = \widetilde{\mathbb{E}}_t \left[\lim_{n\to\infty} \prod_{i=0}^{n} e^{-\frac{\mathbb{1}' b_{t+i\delta,T} \rho b_{t+i\delta,T} \mathbb{1}}{2} \delta - \mathbb{1}' b_{t+i\delta,T} Z_i \sqrt{\delta}} \right]$$

$$\begin{aligned}
&= \lim_{n\to\infty} \widetilde{\mathbb{E}}_t \left[\prod_{i=0}^{n} \left(1 + \left(-\frac{\mathbb{1}' b_{t+i\delta,T} \rho b_{t+i\delta,T} \mathbb{1}}{2} \delta - \mathbb{1}' b_{t+i\delta,T} Z_i \sqrt{\delta} \right) \right.\right. \\
&\qquad \left.\left. + \frac{1}{2} \left(-\frac{\mathbb{1}' b_{t+i\delta,T} \rho b_{t+i\delta,T} \mathbb{1}}{2} \delta - \mathbb{1}' b_{t+i\delta,T} Z_i \sqrt{\delta} \right)^2 + \cdots \right) \right] \\
&= \lim_{n\to\infty} \widetilde{\mathbb{E}}_t \left[\prod_{i=0}^{n} \left(1 - \frac{\mathbb{1}' b_{t+i\delta,T} \rho b_{t+i\delta,T} \mathbb{1}}{2} \delta - \mathbb{1}' b_{t+i\delta,T} Z_i \sqrt{\delta} + \frac{\mathbb{1}' b_{t+i\delta,T} Z_i Z_i' b_{t+i\delta,T} \mathbb{1}}{2} \delta \right.\right. \\
&\qquad \left.\left. + O\left(\delta^{\frac{3}{2}}\right) \right) \right] \\
&= \lim_{n\to\infty} \widetilde{\mathbb{E}}_t \left[1 + \sum_{i=0}^{n} \left(-\frac{\mathbb{1}' b_{t+i\delta,T} \rho b_{t+i\delta,T} \mathbb{1}}{2} \delta - \mathbb{1}' b_{t+i\delta,T} Z_i \sqrt{\delta} + \frac{\mathbb{1}' b_{t+i\delta,T} Z_i Z_i' b_{t+i\delta,T} \mathbb{1}}{2} \delta \right) \right. \\
&\qquad \left. + \sum_{i=0}^{n} \sum_{j>i}^{n} \mathbb{1}' b_{t+i\delta,T} Z_i Z_j' b_{t+j\delta,T} \mathbb{1} \delta + O\left(\delta^{\frac{3}{2}}\right) \right] \\
&= 1
\end{aligned} \tag{235}$$

where $Z_i \sim \mathcal{N}(0,\rho)$ is a series of independent and identically distributed standard normal random variables possessing the properties

$$\widetilde{\mathbb{E}}_t\left[Z_i Z_j'\right] = \begin{cases}\rho & \text{if } i = j \\ 0 & \text{if } i \neq j\end{cases} \quad \text{and} \quad \widetilde{\mathbb{E}}_t\left[b_{t+i\delta,T} Z_j\right]\begin{cases}\neq 0 & \text{if } i > j \\ = 0 & \text{if } i \leq j\end{cases} \tag{236}$$

A look-alike proof can be found in [30]. However, it is practically distinct from the one shown above, as the T variable appears in both the integrand and the upper bound of the integral. Nevertheless, the idea still applies.

The bond price dynamics, presented in (223), can also be obtained from the bond price. First taking logarithm on both sides gives

$$\log P_{t,T} = \log\frac{P_{s,T}}{P_{s,t}} - \int_s^t \mathbb{1}' \frac{b_{u,T}\rho b_{u,T} - b_{u,t}\rho b_{u,t}}{2}\mathbb{1} du - \int_s^t \mathbb{1}'\left(b_{u,T} - b_{u,t}\right) d\widetilde{W}_u \tag{237}$$

We then differentiate both sides with respect to t using Ito's lemma and substitute with r_t in (226)

$$\begin{aligned}
&\frac{dP_{t,T}}{P_{t,T}} - \frac{1}{2}\frac{dP_{t,T}}{P_{t,T}}\frac{dP_{t,T}}{P_{t,T}} \\
&= f_{s,t}dt - \frac{1}{2}\mathbb{1}' b_{t,T}\rho b_{t,T}\mathbb{1} dt + \mathbb{1}'\int_s^t \beta_{u,t}\rho b_{u,t} du\,\mathbb{1} dt - \mathbb{1}' b_{t,T} d\widetilde{W}_t + \mathbb{1}'\int_s^t \beta_{u,t} d\widetilde{W}_u \\
&= r_t dt - \frac{1}{2}\mathbb{1}' b_{t,T}\rho b_{t,T}\mathbb{1} dt - \mathbb{1}' b_{t,T} d\widetilde{W}_t
\end{aligned} \tag{238}$$

The respective drift and diffusion terms are matched to obtain the bond price dynamics

$$\frac{dP_{t,T}}{P_{t,T}} = r_t dt - \mathbb{1}' b_{t,T} d\widetilde{W}_t \tag{239}$$

This is identical to (223), as expected.

Price of a forward bond, defined as $P_{t,T,V} = \mathbb{E}_t^T\left[P_{T,V}\right] = P_{t,V}/P_{t,T}$ for $s < t < T < V$, can be computed from (232), and then subsequently expressed under $\mathbb{Q}^T$ via (221)

$$\begin{aligned}\frac{P_{t,T,V}}{P_{s,T,V}}&=\exp\left(-\int_s^t\mathbb{1}'\frac{b_{u,V}\rho b_{u,V}-b_{u,T}\rho b_{u,T}}{2}\mathbb{1}du-\int_s^t\mathbb{1}'\left(b_{u,V}-b_{u,T}\right)d\widetilde{W}_u\right)\\&=\exp\left(-\int_s^t\mathbb{1}'\frac{\left(b_{u,V}-b_{u,T}\right)\rho\left(b_{u,V}-b_{u,T}\right)}{2}\mathbb{1}du-\int_s^t\mathbb{1}'\left(b_{u,V}-b_{u,T}\right)dW_u^T\right)\end{aligned}\tag{240}$$

$$\frac{dP_{t,T,V}}{P_{t,T,V}}=-\mathbb{1}'\left(b_{t,V}-b_{t,T}\right)\rho b_{t,T}\mathbb{1}dt-\mathbb{1}'\left(b_{t,V}-b_{t,T}\right)d\widetilde{W}_t=-\mathbb{1}'\left(b_{t,V}-b_{t,T}\right)dW_t^T$$

Following the proof (235), it can be shown that the forward bond is a $\mathbb{Q}^T$ martingale. In fact, this can be directly inferred from (22) using bond volatilities $-b_{t,T}$ and $-b_{t,V}$. In a more specific case, if $b_{t,T}$ is deterministic, the forward bond becomes a lognormal martingale expressed in a $\mathbb{Q}^T$ joint normal Z^T with a total variance $\xi^2_{s,t,T,V}$ given in (229)

$$P_{t,T,V}=P_{s,T,V}\exp\left(-\frac{\mathbb{1}'\xi^2_{s,t,T,V}\mathbb{1}}{2}-\mathbb{1}'Z^T\right),\qquad Z^T=\int_s^t\left(b_{u,V}-b_{u,T}\right)dW_u^T\sim\mathcal{N}\left(0,\xi^2_{s,t,T,V}\right)\tag{241}$$

7.4. Caplet and Floorlet

Let us assume the Libor rate $L_{U,V}$ is fixed at T such that $T\oplus\Delta_s=U$, we may write the caplet value at start time s from (78) as

$$\begin{aligned}V^{\text{CPL}}_{s,T,U,V}&=P_{s,V}\mathbb{E}_s^V\left[\tau\left(L_{U,V}-K\right)^+\right]=P_{s,V}\mathbb{E}_s^V\left[\tau\left(\frac{P_{T,U}-P_{T,V}}{\tau P_{T,V}}-K\right)^+\right]\\&=P_{s,V}\mathbb{E}_s^V\left[\frac{\left(P_{T,U}-(1+K\tau)P_{T,V}\right)^+}{P_{T,V}}\right]=P_{s,T}\mathbb{E}_s^T\left[\left(P_{T,U}-(1+K\tau)P_{T,V}\right)^+\right]\end{aligned}\tag{242}$$

where $\tau=V-U$. If we further ignore the spot lag and assume $U=T$, the caplet/floorlet can be treated as a put/call option on a spot zero-coupon bond $P_{T,V}$ (equivalent to $P_{T,T,V}$), that is

$$V^{\text{CPL}}_{s,T,T,V}=P_{s,T}\mathbb{E}_s^T\left[\left(1-(1+K\tau)P_{T,V}\right)^+\right]=(1+K\tau)P_{s,T}\mathbb{E}_s^T\left[\left(\frac{1}{1+K\tau}-P_{T,V}\right)^+\right]\tag{243}$$

In general, the distribution of $P_{t,T,V}$ is unknown. However, if $b_{t,T}$ is deterministic, the $P_{t,T,V}$ becomes a $\mathbb{Q}^T$ lognormal martingale, the caplet price can then be calculated by *Black* formula (81) using total variance $\xi^2_{s,T,T,V}$ defined in (241)

$$V_{s,T,T,V}^{\mathrm{CPL}} = (1+K\tau)P_{s,T}\mathfrak{B}\left(\frac{1}{1+K\tau}, P_{s,T,V}, \xi_{s,T,T,V}^2, -1\right) \tag{244}$$

7.5. Swaption

From (88) we see that payer swaption entered at s and matured at T for $T \oplus \Delta_s = T_a$ can be valued (after change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$) by

$$\begin{aligned} V_{s,T,a,b}^{\mathrm{PS}} &= P_{s,T}\mathbb{E}_s^T\left[\left(\sum_{i=a+1}^{b} P_{T,i}\tau_i\left(L_{T,i}-K\right)\right)^+\right] = P_{s,T}\mathbb{E}_s^T\left[\left(P_{T,a}-P_{T,b}-K\sum_{i=a+1}^{b} P_{T,i}\tau_i\right)^+\right] \\ &= P_{s,T}\mathbb{E}_s^T\left[\left(\sum_{i=a}^{b} c_i P_{T,i}\right)^+\right], \qquad c_i = \begin{cases} 1 & \text{if } i=a \\ -\tau_i K & \text{if } a+1 \le i \le b-1 \\ -1-\tau_i K & \text{if } i=b \end{cases} \end{aligned} \tag{245}$$

The exact distribution of the summation is not well defined. Nevertheless, in certain models, the swaption price can still be tractable with special treatments. This will be discussed in Section 8.4.2.

8. Short Rate Models

In this section, we will firstly provide an overview of affine term structure models, a broad class that many short rate models belong to. Then we will focus on the derivation, calibration and application of the Hull White model, a Gaussian affine term structure model widely used in industry.

8.1. Arbitrage-free Bond Pricing

Suppose there is a short rate r_t following a stochastic process in a general form

$$dr_t = \mu_t dt + \mathbb{1}'\sigma_t dW_t \tag{246}$$

where dW_t is an $n \times 1$ Brownian motion under physical measure, scalar μ_t and diagonal-matrix-valued σ_t are instantaneous drift and volatility process for r_t respectively. Assuming the price of a bond $P_{t,T}$ depends only on the spot rate r_t, current time t and bond maturity T, then Ito's lemma gives the bond dynamics as

$$\begin{aligned} dP_{t,T} &= \frac{\partial P_{t,T}}{\partial t} dt + \frac{\partial P_{t,T}}{\partial r} dr + \frac{1}{2} dr' \frac{\partial^2 P_{t,T}}{\partial r^2} dr \\ &= \frac{\partial P_{t,T}}{\partial t} dt + \mu_t \frac{\partial P_{t,T}}{\partial r} + \frac{\partial P_{t,T}}{\partial r} \mathbb{1}'\sigma_t dW_t + \frac{1}{2} \frac{\partial^2 P_{t,T}}{\partial r^2} dW_t' \sigma_t \mathbb{1}\mathbb{1}'\sigma_t dW_t \\ &= \left(\frac{\partial P_{t,T}}{\partial t} + \mu_t \frac{\partial P_{t,T}}{\partial r} + \frac{\mathbb{1}'\sigma_t \rho \sigma_t \mathbb{1}}{2} \frac{\partial^2 P_{t,T}}{\partial r^2} \right) dt + \frac{\partial P_{t,T}}{\partial r} \mathbb{1}'\sigma_t dW_t \end{aligned} \tag{247}$$

We assume the bond price follows a geometric Brownian motion with drift $\delta_{t,T}$ and volatility $\omega_{t,T}$

$$\begin{aligned} &\frac{dP_{t,T}}{P_{t,T}} = \delta_{t,T} dt + \mathbb{1}'\omega_{t,T} dW_t \quad \text{where} \\ &\delta_{t,T} = \frac{1}{P_{t,T}} \left(\frac{\partial P_{t,T}}{\partial t} + \mu_t \frac{\partial P_{t,T}}{\partial r} + \frac{\mathbb{1}'\sigma_t \rho \sigma_t \mathbb{1}}{2} \frac{\partial^2 P_{t,T}}{\partial r^2} \right), \qquad \omega_{t,T} = \frac{1}{P_{t,T}} \frac{\partial P_{t,T}}{\partial r} \sigma_t \end{aligned} \tag{248}$$

Since the short rate is not a tradable asset, it cannot be used to hedge with the bond, instead we try to hedge bonds of different maturities by constructing a portfolio with a long position of dollar value v_T of $P_{t,T}$ and a short position of a set of dollar value $[V_S]_i = v_{S_i} \ \forall\ 1 \le i \le n$ of P_{t,S_i}. The portfolio value Π_t is given by $\Pi = v_T - V_S'\mathbb{1}$ and the instantaneous change in portfolio value is

$$
\begin{aligned}
&d\Pi = v_T\left(\delta_{t,T}dt + \mathbb{1}'\omega_{t,T}dW_t\right) - V_S'\left(\Delta_{t,S}dt + \Omega_{t,S}dW_t\right) \\
&\text{where} \qquad \left[\Delta_{t,S}\right]_i = \delta_{t,S_i}, \qquad \left[\Omega_{t,S}\right]_{i,j} = \left[\omega_{t,S_i}\right]_{j,j}
\end{aligned} \tag{249}
$$

The $\Delta_{t,S}$ is an $n \times 1$ vector and the $\Omega_{t,S}$ is an $n \times n$ matrix whose i-th row is $\mathbb{1}'\omega_{t,S_i}$. If we choose the dollar values such that

$$
v_T\mathbb{1}'\omega_{t,T} = V_S'\Omega_{t,S} \quad \Longrightarrow \quad V_S' = v_T\mathbb{1}'\omega_{t,T}\Omega_{t,S}^{-1} \tag{250}
$$

Then the stochastic term in (249) vanishes and the equation becomes

$$
\frac{d\Pi}{\Pi} = \frac{v_T\delta_{t,T} - V_S'\Delta_{t,S}}{v_T - V_S'\mathbb{1}}dt = r_t dt \tag{251}
$$

The last equality comes from the fact that the portfolio is instantaneously riskless and thus it must earn the risk-free short rate to avoid arbitrage. This gives

$$
v_T\left(\delta_{t,T} - r_t\right) = V_S'\left(\Delta_{t,S} - r_t\right) = v_T\mathbb{1}'\omega_{t,T}\Omega_{t,S}^{-1}\left(\Delta_{t,S} - r_t\right) \tag{252}
$$

Hence for an arbitrary bond maturity T, there must exist a unique (vector) quantity $\lambda_t = \Omega_{t,S}^{-1}\left(\Delta_{t,S} - r_t\right)$, which is independent of T, such that

$$
\mathbb{1}'\omega_{t,T}\lambda_t = \delta_{t,T} - r_t \tag{253}
$$

The λ_t, same as in (217), is called *market price of interest rate risk*, as it gives the extra increase in expected instantaneous rate of return on a bond per an additional unit of risk. Using the expressions in (248), we arrive at the governing partial differential equation for the bond price

$$
\begin{aligned}
&\frac{1}{P_{t,T}}\left(\frac{\partial P_{t,T}}{\partial t} + \mu_t\frac{\partial P_{t,T}}{\partial r} + \frac{\mathbb{1}'\sigma_t\rho\sigma_t\mathbb{1}}{2}\frac{\partial^2 P_{t,T}}{\partial r^2}\right) - r_t = \frac{1}{P_{t,T}}\frac{\partial P_{t,T}}{\partial r}\mathbb{1}'\sigma_t\lambda_t \\
&\Longrightarrow \frac{\partial P_{t,T}}{\partial t} + \left(\mu_t - \mathbb{1}'\sigma_t\lambda_t\right)\frac{\partial P_{t,T}}{\partial r} + \frac{\mathbb{1}'\sigma_t\rho\sigma_t\mathbb{1}}{2}\frac{\partial^2 P_{t,T}}{\partial r^2} - r_t P_{t,T} = 0
\end{aligned} \tag{254}
$$

The solution of the bond price is

$$
P_{t,T} = \mathbb{E}_t\left[\exp\left(-\int_t^T r_u du - \frac{1}{2}\int_t^T \lambda_u'\rho^{-1}\lambda_u du - \int_t^T \lambda_u'\rho^{-1}dW_u\right)\right] \tag{255}
$$

To show the claim, we define an auxiliary function for $t > s$

$$V_{s,t} = \exp\left(-\int_s^t r_u du - \frac{1}{2}\int_s^t \lambda_u' \rho^{-1} \lambda_u du - \int_s^t \lambda_u' \rho^{-1} dW_u\right) \tag{256}$$

Applying Ito differential rule to $V_{s,t}$ with respect to t, we have

$$\frac{dV_{s,t}}{V_{s,t}} = -r_t dt - \frac{\lambda_t' \rho^{-1} \lambda_t}{2} dt - \lambda_t' \rho^{-1} dW_t + \frac{\lambda_t' \rho^{-1} \lambda_t}{2} dt = -r_t dt - \lambda_t' \rho^{-1} dW_t \tag{257}$$

and then to $V_{s,t} P_{t,T}$, we get

$$\begin{aligned} \frac{d\left(V_{s,t} P_{t,T}\right)}{V_{s,t} P_{t,T}} &= \frac{dV_{s,t}}{V_{s,t}} + \frac{dP_{t,T}}{P_{t,T}} + \frac{dV_{s,t} dP_{t,T}}{V_{s,t} P_{t,T}} \\ &= -r_t dt - \lambda_t' \rho^{-1} dW_t + \delta_{t,T} dt + \mathbb{1}' \omega_{t,T} dW_t - \mathbb{1}' \omega_{t,T} \lambda_t dt \\ &= \left(\delta_{t,T} - r_t - \mathbb{1}' \omega_{t,T} \lambda_t\right) d\tau + \left(\mathbb{1}' \omega_{t,T} - \lambda_t' \rho^{-1}\right) dW_t \\ &= \left(\mathbb{1}' \omega_{t,T} - \lambda_t' \rho^{-1}\right) dW_t \end{aligned} \tag{258}$$

The (258) shows that the $V_{t,\tau} P_{\tau,T}$ process is a martingale under physical measure, and since $V_{t,t} = 1$ and $P_{T,T} = 1$, we must have

$$P_{t,T} = \mathbb{E}_t\left[V_{t,T}\right] = \mathbb{E}_t\left[\exp\left(-\int_t^T r_u\, du - \frac{1}{2}\int_t^T \lambda_u' \rho^{-1} \lambda_u\, du - \int_t^T \lambda_u' \rho^{-1} dW_u\right)\right] \tag{259}$$

If we let $\lambda_t = \rho \theta_t$ then according to (13) we have

$$Z_t = \exp\left(-\frac{1}{2}\int_s^t \lambda_u' \rho^{-1} \lambda_u du - \int_s^t \lambda_u' \rho^{-1} dW_u\right) \quad \text{and} \quad d\widetilde{W}_t = dW_t + \lambda_t dt \tag{260}$$

and the change of measure shows

$$P_{t,T} = \mathbb{E}_t\left[V_{t,T}\right] = \mathbb{E}_t\left[\exp\left(-\int_t^T r_u\, du\right)\frac{Z_T}{Z_t}\right] = \widetilde{\mathbb{E}}_t\left[\exp\left(-\int_t^T r_u\, du\right)\right] \tag{261}$$

This is the bond price under risk neutral measure. It follows exactly the arbitrage free pricing theory. Since the change from physical measure to risk neutral measure (or vice versa) can be easily achieved by including a vector of *market price of risk* λ_t such that

$$d\widetilde{W}_t = dW_t + \lambda_t dt \tag{262}$$

we put our focus on the rate dynamics under risk neutral measure for its simplicity.

8.2. Affine Term Structure

By (227), the short rate can be expressed as an affine function of state vector x_t, i.e., $r_t = f_{s,t} + \mathbb{1}'x_t$. Suppose under risk neutral measure, the state vector x_t follows a diffusion process governed by a general SDE

$$dx_t = \alpha_t dt + \sigma_t d\widetilde{W}_t \tag{263}$$

with

$$\underset{n\times 1}{\alpha_t} = \begin{bmatrix} \vdots \\ \alpha_{i;t} \\ \vdots \end{bmatrix}, \qquad \underset{n\times n}{\sigma_t} = \text{Diag}\begin{bmatrix} \vdots \\ \sigma_{i;t} \\ \vdots \end{bmatrix}, \qquad \underset{n\times 1}{d\widetilde{W}_t} = \begin{bmatrix} \vdots \\ d\widetilde{W}_{i;t} \\ \vdots \end{bmatrix}, \qquad d\widetilde{W}_t d\widetilde{W}_t' = \underset{n\times n}{\rho}\, dt \tag{264}$$

the short rate must admit dynamics like

$$dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}'\alpha_t\right)dt + \mathbb{1}'\sigma_t d\widetilde{W}_t \tag{265}$$

If the model can produce zero coupon bond prices in an *exponential-affine* form

$$P_{t,T} = \exp\left(-\underset{1\times 1}{A_{t,T}} - \mathbb{1}'\underset{n\times n}{B_{t,T}} x_t\right) \quad \forall\, 0 \le t \le T \tag{266}$$

with scalar $A_{t,T}$ and diagonal matrix $B_{t,T}$ being deterministic functions of the time t and the maturity T, we then say it is an affine term structure model (this definition is different from what you typically see in textbooks where the exponent is usually an affine function of the short rate r_t rather than the latent variable x_t, it's easy to show that the two definitions are algebraically equivalent). The tractability in bond price is the main advantage of affine models. In fact, by design, many short rate models admit an affine term structure. To observe this, we first differentiate the bond price via Ito's lemma

$$\begin{aligned} dP_{t,T} &= \frac{\partial P_{t,T}}{\partial t}dt + \frac{\partial P_{t,T}}{\partial x}dx_t + \frac{1}{2}dx_t'\frac{\partial^2 P_{t,T}}{\partial x^2}dx_t \\ \Rightarrow \frac{dP_{t,T}}{P_{t,T}} &= -\frac{\partial A_{t,T}}{\partial t}dt - \mathbb{1}'\frac{\partial B_{t,T}}{\partial t}x_t dt - \mathbb{1}'B_{t,T}dx_t + \frac{1}{2}d\widetilde{W}_t'\sigma_t B_{t,T}\mathbb{1}\mathbb{1}'B_{t,T}\sigma_t d\widetilde{W}_t \\ \Rightarrow \frac{dP_{t,T}}{P_{t,T}} &= \left(\frac{\mathbb{1}'B_{t,T}\sigma_t\rho\sigma_t B_{t,T}\mathbb{1}}{2} - \frac{\partial A_{t,T}}{\partial t} - \mathbb{1}'\frac{\partial B_{t,T}}{\partial t}x_t - \mathbb{1}'B_{t,T}\alpha_t\right)dt - \mathbb{1}'B_{t,T}\sigma_t d\widetilde{W}_t \end{aligned} \tag{267}$$

Since the bond is a market tradable asset, in order to avoid arbitrage, the drift term in the above equation must equal to r_t under risk neutral measure $\mathbb{Q}$, that is

$$\frac{\mathbb{1}' B_{t,T}\sigma_t\rho\sigma_t B_{t,T}\mathbb{1}}{2} - \frac{\partial A_{t,T}}{\partial t} - \mathbb{1}'\frac{\partial B_{t,T}}{\partial t}x_t - \mathbb{1}' B_{t,T}\alpha_t - f_{s,t} - \mathbb{1}' x_t = 0 \tag{268}$$

If one can solve for the $A_{t,T}$ and $B_{t,T}$ (which are independent of x_t), the model admits an affine term structure. A sufficient condition [31] [32] for this would be that both the drift α_t and the sum of covariance matrix columns $B_{t,T}\sigma_t\rho\sigma_t B_{t,T}\mathbb{1}$ are affine functions of x_t (i.e., $\alpha_t = a_t + b_t x_t$ and $B_{t,T}\sigma_t\rho\sigma_t B_{t,T}\mathbb{1} = p_t + q_t x_t$ for vector-valued functions a_t, p_t and diagonal-matrix-valued functions b_t, q_t). This assumption transforms (268) into

$$-\frac{\partial A_{t,T}}{\partial t} + \frac{\mathbb{1}' p_t}{2} - f_{s,t} - \mathbb{1}' B_{t,T}a_t - \mathbb{1}'\left(\frac{\partial B_{t,T}}{\partial t} + B_{t,T}b_t - \frac{q_t}{2} + I\right)x_t = 0 \tag{269}$$

with I denoting an identity matrix. Because (269) must hold for all x_t, its coefficient must vanish, we can derive the following two ODE's

$$\begin{aligned} &\frac{\partial B_{t,T}}{\partial t} + B_{t,T}b_t - \frac{q_t}{2} + I = 0, \qquad B_{T,T} = 0 \\ &\frac{\partial A_{t,T}}{\partial t} - \frac{\mathbb{1}' p_t}{2} + \mathbb{1}' B_{t,T}a_t + f_{s,t} = 0, \qquad A_{T,T} = 0 \end{aligned} \tag{270}$$

The terminal conditions $A_{T,T} = 0$ and $B_{T,T} = 0$ are implied by the fact that the bond value upon maturity at T is always at 1 regardless of the value of x_T. The equation for $B_{t,T}$ does not involve $A_{t,T}$, which can be solved first, then using the solution of $B_{t,T}$ to solve for $A_{t,T}$ by integration. Both $A_{t,T}$ and $B_{t,T}$ are known and are independent of x_t. In a single factor model (i.e., x_t is scalar-valued), the covariance matrix $B_{t,T}\sigma_t\rho\sigma_t B_{t,T}$ degenerates into a scalar. The sufficient condition [33] simplifies to: both drift term α_t and variance term σ_t^2 of x_t are affine functions of x_t such that $\alpha_t = a_t + b_t x_t$ and $\sigma_t^2 = p_t + q_t x_t$. Under this condition, the (268) reduces to

$$-\frac{\partial A_{t,T}}{\partial t} + \frac{1}{2}B_{t,T}^2 p_t - f_{s,t} - B_{t,T}a_t - \left(\frac{\partial B_{t,T}}{\partial t} + B_{t,T}b_t - \frac{1}{2}B_{t,T}^2 q_t + 1\right)x_t = 0 \tag{271}$$

and therefore gives the following two ODE's that can be solved in the same manner

$$\frac{\partial B_{t,T}}{\partial t} + B_{t,T} b_t - \frac{1}{2} B_{t,T}^2 q_t + 1 = 0, \qquad B_{T,T} = 0$$
$$\frac{\partial A_{t,T}}{\partial t} - \frac{1}{2} B_{t,T}^2 p_t + B_{t,T} a_t + f_{s,t} = 0, \qquad A_{T,T} = 0 \tag{272}$$

Every term structure model, including affine models, driven by Brownian motion is an HJM model. This is because, in any of such models, there are forward rates, the drift and diffusion of the forward rates must satisfy the condition in (220) for a risk neutral measure to exist, which rules out arbitrage. To see this, we first drive the forward rate using definition (40) and bond price (266)

$$f_{t,T} = -\frac{\partial \log P_{t,T}}{\partial T} = \frac{\partial A_{t,T}}{\partial T} + \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} x_t \tag{273}$$

The forward rate dynamics is then derived

$$\begin{aligned} df_{t,T} &= \frac{\partial^2 A_{t,T}}{\partial T \partial t} dt + \mathbb{1}' \frac{\partial^2 B_{t,T}}{\partial T \partial t} x_t dt + \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} dx_t \\ &= \left(\frac{\partial^2 A_{t,T}}{\partial T \partial t} + \mathbb{1}' \frac{\partial^2 B_{t,T}}{\partial T \partial t} x_t + \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} \alpha_t \right) dt + \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} \sigma_t d\widetilde{W}_t \\ &= \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} \sigma_t \rho \sigma_t B_{t,T} \mathbb{1} dt + \mathbb{1}' \frac{\partial B_{t,T}}{\partial T} \sigma_t d\widetilde{W}_t \end{aligned} \tag{274}$$

where the last equality is obtained by applying partial derivative $\partial/\partial T$ to (268). With bond volatility $b_{t,T} = B_{t,T}\sigma_t$, we can see that the forward rate dynamics conforms to the HJM no-arbitrage condition (223) as expected. In fact, this equivalency is assured because no-arbitrage condition is unique.

8.3. Quasi-Gaussian Model

8.3.1. General Form

8.3.1.1. *Stochastic Process*

Quasi-Gaussian model, also known as Cheyette model [34] [35], was named by Jamshidian [36] for a class of Markovian HJM models with a separable volatility structure. It is a special case of the HJM framework. In Quasi-Gaussian model, the forward rate volatility function $\beta_{t,T}$, as in (209), is assumed to be in a separable form, i.e., a product of a maturity dependent function and a time dependent function

$$\beta_{t,T} = \frac{\sigma_t}{\lambda_t}\lambda_T \tag{275}$$

where both σ_t and λ_t are $n \times n$ diagonal-matrix-valued functions. Given such volatility specification, the bond volatility (214) transforms into

$$b_{t,T} = \int_t^T \beta_{t,u} du = \frac{\sigma_t}{\lambda_t}\Lambda_{t,T}, \qquad \Lambda_{t,T} = \int_t^T \lambda_v dv \tag{276}$$

Meanwhile, the stochastic processes x_t and y_t in (228) become

$$\begin{aligned} x_t &= \lambda_t \int_s^t \frac{\sigma_u}{\lambda_u}\rho\frac{\sigma_u}{\lambda_u}\Lambda_{u,t} du\, \mathbb{1} + \lambda_t \int_s^t \frac{\sigma_u}{\lambda_u} d\widetilde{W}_u, \qquad dx_t = \left(y_t \mathbb{1} + \frac{\partial \log \lambda_t}{\partial t} x_t\right) dt + \sigma_t d\widetilde{W}_t \\ y_t &= \lambda_t \int_s^t \frac{\sigma_u}{\lambda_u}\rho\frac{\sigma_u}{\lambda_u} du\, \lambda_t, \qquad dy_t = \left(\frac{\partial \log \lambda_t}{\partial t} y_t + y_t \frac{\partial \log \lambda_t}{\partial t} + \sigma_t \rho \sigma_t\right) dt \end{aligned} \tag{277}$$

The joint process of x_t and y_t are now Markovian, and hence can be expressed in a recursive form for $s < v < t$

$$x_t = x_{t|s} = \frac{\lambda_t}{\lambda_v}\left(x_{v|s} + y_{v|s}\frac{\Lambda_{v,t}}{\lambda_v}\mathbb{1}\right) + x_{t|v}, \qquad y_t = y_{t|s} = \frac{\lambda_t}{\lambda_v} y_{v|s} \frac{\lambda_t}{\lambda_v} + y_{t|v} \tag{278}$$

8.3.1.2. *Forward Rate and Short Rate*

The forward rate (225) and its dynamics (220) can be expressed in terms of the joint Markovian process x_t and y_t (277)

$$\begin{aligned} f_{t,T} &= f_{s,T} + \mathbb{1}' \int_s^t \beta_{u,T}\rho b_{u,T} du\, \mathbb{1} + \mathbb{1}' \int_s^t \beta_{u,T} d\widetilde{W}_u = f_{s,T} + \mathbb{1}' \frac{\lambda_T}{\lambda_t}\left(y_t \frac{\Lambda_{t,T}}{\lambda_t}\mathbb{1} + x_t\right) \\ df_{t,T} &= \mathbb{1}'\beta_{t,T}\rho b_{t,T}\mathbb{1} dt + \mathbb{1}'\beta_{t,T} d\widetilde{W}_t = \mathbb{1}'\lambda_T \frac{\sigma_t}{\lambda_t}\rho\frac{\sigma_t}{\lambda_t}\Lambda_{t,T}\mathbb{1} dt + \mathbb{1}'\lambda_T \frac{\sigma_t}{\lambda_t} d\widetilde{W}_t \end{aligned} \tag{279}$$

Similarly, we have the short rate and its dynamics (226) expressed as

$$r_t = f_{s,t} + \mathbb{1}' x_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}' \frac{\partial \log \lambda_t}{\partial t} x_t + \mathbb{1}' y_t \mathbb{1}\right) dt + \mathbb{1}'\sigma_t d\widetilde{W}_t \tag{280}$$

Note that in (275), the λ_t is usually taken as a simple time-dependent deterministic function. The σ_t can be stochastic. For example, it can be a local volatility $\sigma_t \equiv \sigma(t, x_t, y_t)$, or a stochastic volatility $\sigma_t \equiv \sigma(t, z_t)$ driven by a separate stochastic factor z_t, or a stochastic local volatility $\sigma_t \equiv \sigma(t, x_t, y_t, z_t)$. Since the σ_t is stochastic, the x_t and consequentially the short rate r_t are unlikely normally distributed (hence the model is called Quasi-Gaussian model). In contrast, if σ_t is deterministic, the short rate r_t follows a normal distribution, and the model degenerates into a Gaussian model.

8.3.1.3. *Zero Coupon Bond*

The zero-coupon bond (232) can be expressed as an *exponential-affine* function of the Markovian state variables x_t and y_t in (277)

$$P_{t,T} = \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{1}{2}\mathbb{1}'\frac{\Lambda_{t,T}}{\lambda_t} y_t \frac{\Lambda_{t,T}}{\lambda_t}\mathbb{1} - \mathbb{1}'\frac{\Lambda_{t,T}}{\lambda_t} x_t\right), \qquad \frac{dP_{t,T}}{P_{t,T}} = r_t dt - \mathbb{1}'\frac{\Lambda_{t,T}}{\lambda_t}\sigma_t d\widetilde{W}_t \tag{281}$$

It should not be confused with the sufficient condition stated in section 8.2. Here the affine function is on the joint state variables x_t and y_t, whereas the aforesaid sufficient condition is imposed on the process x_t only in order to admit an affine term structure.

The forward bond and its dynamics (240) transform into

$$\begin{aligned}
\frac{P_{t,T,V}}{P_{s,T,V}} &= \exp\left(-\frac{1}{2}\mathbb{1}'\left(\frac{\Lambda_{t,V}}{\lambda_t} y_t \frac{\Lambda_{t,V}}{\lambda_t} - \frac{\Lambda_{t,T}}{\lambda_t} y_t \frac{\Lambda_{t,T}}{\lambda_t}\right)\mathbb{1} - \mathbb{1}'\frac{\Lambda_{T,V}}{\lambda_t} x_t\right) \\
&= \exp\left(-\frac{1}{2}\mathbb{1}'\frac{\Lambda_{T,V}}{\lambda_t}\, y_t \frac{\Lambda_{T,V}}{\lambda_t}\mathbb{1} - \mathbb{1}'\Lambda_{T,V}\int_s^t \frac{\sigma_u}{\lambda_u}\rho\frac{\sigma_u}{\lambda_u}\Lambda_{u,T}\mathbb{1}du - \mathbb{1}'\Lambda_{T,V}\int_s^t \frac{\sigma_u}{\lambda_u} d\widetilde{W}_u\right) \\
&= \exp\left(-\frac{1}{2}\mathbb{1}'\frac{\Lambda_{T,V}}{\lambda_t}\, y_t \frac{\Lambda_{T,V}}{\lambda_t}\mathbb{1} - \mathbb{1}'\Lambda_{T,V}\int_s^t \frac{\sigma_u}{\lambda_u} dW_u^T\right) \\
\frac{dP_{t,T,V}}{P_{t,T,V}} &= -\mathbb{1}'\Lambda_{T,V}\frac{\sigma_t}{\lambda_t}\rho\frac{\sigma_t}{\lambda_t}\Lambda_{t,T}\mathbb{1}dt - \mathbb{1}'\Lambda_{T,V}\frac{\sigma_t}{\lambda_t}d\widetilde{W}_t = -\mathbb{1}'\Lambda_{T,V}\frac{\sigma_t}{\lambda_t}dW_t^T
\end{aligned} \tag{282}$$

where we have applied the change of measure (221)

$$dW_u^T = d\widetilde{W}_u + \rho\frac{\sigma_u}{\lambda_u}\Lambda_{u,T}\mathbb{1}du \tag{283}$$

8.3.2. Mean Reversion

It is often convenient to assume λ_t to be a simple function of time, for example

$$\frac{\partial \log \lambda_t}{\partial t} = -\kappa_t \Longrightarrow \lambda_t = \exp\left(-\int_s^t \kappa_u du\right) \equiv E_{s,t} \quad \forall \;\; s < t \tag{284}$$

where κ_t, called mean reversion rate, is an $n \times n$ diagonal-matrix-valued function of time. Given this assumption, the forward rate volatility (275) becomes

$$\beta_{t,T} = \frac{\sigma_t}{\lambda_t}\lambda_T = E_{t,T}\sigma_t, \qquad \underset{n\times n}{E_{t,T}} = \mathrm{Diag}\begin{bmatrix} \vdots \\ E_{i;t,T} \\ \vdots \end{bmatrix}, \qquad E_{i;t,T} \equiv \exp\left(-\int_t^T \kappa_{i;u} du\right) \tag{285}$$

With such volatility specification, the bond volatility (276) transforms into

$$b_{t,T} = \int_t^T \beta_{t,u} du = \int_t^T E_{t,u}\sigma_t du = B_{t,T}\sigma_t, \qquad B_{t,T} = \int_t^T E_{t,u} du \tag{286}$$

and the following identities are often used

$$\Lambda_{t,T} = \int_t^T E_{s,v} dv\,, \qquad \frac{\Lambda_{t,T}}{\lambda_t} = \int_t^T E_{t,v} dv = B_{t,T}, \qquad \frac{\Lambda_{T,V}}{\lambda_t} = B_{t,V} - B_{t,T} = B_{T,V}E_{t,T} \tag{287}$$

Based on the last equality in (288), we can also derive using definitions in (229) that

$$\begin{aligned} &\psi_{s,t,V,V} - \psi_{s,t,T,T} = B_{T,V}\,\varphi_{s,t,T,T}B_{T,V} + 2B_{T,V}\chi_{s,t,T,T} \\ &\chi_{s,t,T,V} - \chi_{s,t,T,T} = B_{T,V}\varphi_{s,t,T,T} \end{aligned} \tag{288}$$

8.3.2.1. *Stochastic Process*

With $\lambda_t = E_{s,t}$, the joint Markovian process x_t and y_t in (277) reduce into

$$\begin{aligned} &x_t = \int_s^t E_{u,t}\sigma_u\rho\sigma_u B_{u,t} du\, \mathbb{1} + \int_s^t E_{u,t}\sigma_u d\widetilde{W}_u\,, \qquad dx_t = (y_t\mathbb{1} - \kappa_t x_t)dt + \sigma_t d\widetilde{W}_t \\ &y_t = \varphi_{s,t,t,t} = \int_s^t E_{u,t}\sigma_u\rho\sigma_u E_{u,t} du\,, \qquad dy_t = (-\kappa_t y_t - y_t\kappa_t + \sigma_t\rho\sigma_t)dt \end{aligned} \tag{289}$$

Note that since κ_t is a diagonal matrix and y_t is a (symmetric) variance matrix, the product $\kappa_t y_t$ is different from $y_t\kappa_t$. It can be seen that the x_t exhibits a mean reversion property, and so does the short rate r_t. The recursive definition (278) for $s < v < t$ becomes

$$x_t \equiv x_{t|s} = E_{v,t}\left(x_{v|s} + y_{v|s}B_{v,t}\mathbb{1}\right) + x_{t|v}, \qquad y_t \equiv y_{t|s} = E_{v,t}y_{v|s}E_{v,t} + y_{t|v} \tag{290}$$

8.3.2.2. *Forward Rate and Short Rate*

The forward rate (279) and the short rate (280) thus follow

$$f_{t,T} = f_{s,T} + \mathbb{1}'E_{t,T}y_tB_{t,T}\mathbb{1} + \mathbb{1}'E_{t,T}x_t, \qquad df_{t,T} = \mathbb{1}'E_{t,T}\sigma_t\rho\sigma_tB_{t,T}\mathbb{1}dt + \mathbb{1}'E_{t,T}\sigma_td\widetilde{W}_t$$
$$r_t = f_{s,t} + \mathbb{1}'x_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}'y_t\mathbb{1} - \mathbb{1}'\kappa_tx_t\right)dt + \mathbb{1}'\sigma_td\widetilde{W}_t \tag{291}$$

8.3.2.3. *Zero Coupon Bond*

Lastly, the zero-coupon bond and its dynamics (281) read

$$P_{t,T} = \frac{P_{s,T}}{P_{s,t}}\exp\left(-\frac{1}{2}\mathbb{1}'B_{t,T}y_tB_{t,T}\mathbb{1} - \mathbb{1}'B_{t,T}x_t\right)$$
$$= \frac{P_{s,T}}{P_{s,t}}\exp\left(-\frac{1}{2}\mathbb{1}'B_{t,T}\varphi_{s,t,t,t}B_{t,T}\mathbb{1} - \mathbb{1}'B_{t,T}\chi_{s,t,t,t}\mathbb{1} - \mathbb{1}'B_{t,T}\int_s^t E_{u,t}\sigma_ud\widetilde{W}_u\right) \tag{292}$$
$$\frac{dP_{t,T}}{P_{t,T}} = r_tdt - \mathbb{1}'B_{t,T}\sigma_td\widetilde{W}_t$$

given x_t and y_t in (289) and definitions in (229). The forward bond and its dynamics (282) hence are

$$\frac{P_{t,T,V}}{P_{s,T,V}} = \exp\left(-\mathbb{1}'\frac{B_{t,V}y_tB_{t,V} - B_{t,T}y_tB_{t,T}}{2}\mathbb{1} - \mathbb{1}'B_{T,V}E_{t,T}x_t\right)$$
$$= \exp\left(-\mathbb{1}'\frac{B_{T,V}E_{t,T}y_tE_{t,T}B_{T,V}}{2}\mathbb{1} - \mathbb{1}'B_{T,V}E_{t,T}y_tB_{t,T}\mathbb{1} - \mathbb{1}'B_{T,V}E_{t,T}x_t\right)$$
$$= \exp\left(-\frac{1}{2}\mathbb{1}'B_{T,V}E_{t,T}\,y_tE_{t,T}B_{T,V}\mathbb{1} - \mathbb{1}'B_{T,V}\int_s^t E_{u,T}\sigma_u\rho\sigma_uB_{u,T}\mathbb{1}du - \mathbb{1}'B_{T,V}\int_s^t E_{u,T}\sigma_ud\widetilde{W}_u\right) \tag{293}$$
$$= \exp\left(-\frac{1}{2}\mathbb{1}'B_{T,V}E_{t,T}\,y_tE_{t,T}B_{T,V}\mathbb{1} - \mathbb{1}'B_{T,V}\int_s^t E_{u,T}\sigma_udW_u^T\right)$$
$$\frac{dP_{t,T,V}}{P_{t,T,V}} = -\mathbb{1}'B_{T,V}E_{t,T}\sigma_t\rho\sigma_tB_{t,T}\mathbb{1}dt - \mathbb{1}'B_{T,V}E_{t,T}\sigma_td\widetilde{W}_t = -\mathbb{1}'B_{T,V}E_{t,T}\sigma_tdW_t^T$$

where we have used the change of measure (283)

$$dW_u^T = d\widetilde{W}_u + \rho\sigma_uB_{u,T}\mathbb{1}du \tag{294}$$

8.4. Linear Gaussian Model

Linear Gaussian model is a special case of Quasi-Gaussian model assuming a deterministic volatility process σ_t. The rates and bond dynamics derived in Section 8.3 for Quasi-Gaussian model hence retain the same algebraic forms. For easy reference, the state variables x_t and y_t (289) and the short rate r_t (291) are repeated below

$$
\begin{aligned}
& x_t = \int_s^t E_{u,t}\sigma_u\rho\sigma_u B_{u,t} du\,\mathbb{1} + \int_s^t E_{u,t}\sigma_u d\widetilde{W}_u\,, \qquad dx_t = (y_t\mathbb{1} - \kappa_t x_t)dt + \sigma_t d\widetilde{W}_t \\
& y_t = \int_s^t E_{u,t}\sigma_u\rho\sigma_u E_{u,t} du\,, \qquad dy_t = (-\kappa_t y_t - y_t\kappa_t + \sigma_t\rho\sigma_t)dt \\
& r_t = f_{s,t} + \mathbb{1}' x_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}' y_t \mathbb{1} - \mathbb{1}'\kappa_t x_t\right)dt + \mathbb{1}'\sigma_t d\widetilde{W}_t
\end{aligned}
\tag{295}
$$

Since σ_t is deterministic, the short rate must be normally distributed, and thus it receives the name of Linear Gaussian model. In the following, we summarize a few pricing methods of vanilla interest rate derivatives in the model. Existence of closed-form or semi-closed-form of such pricing formulas is crucial for efficient model calibration. Again, we assume a mean reversion parameter κ_t in the model, the same as in (284) in section 8.3.2.

8.4.1. Caplet and Floorlet

Since Linear Gaussian model conforms to HJM framework and σ_t is deterministic, we can use formula (244) to calculate the price of a caplet on a Libor rate $L_{T,V}$ whose total variance is $\xi^2_{s,T,T,V}$ as defined in (229). Its expression in the context of Linear Gaussian Model becomes

$$
\begin{aligned}
& \xi^2_{s,t,T,V} = \int_s^t \mathbb{1}'\big(B_{u,V} - B_{u,T}\big)\sigma_u\rho\sigma_u\big(B_{u,V} - B_{u,T}\big)\mathbb{1} du = \mathbb{1}' B_{T,V}\varphi_{s,t,T,T} B_{T,V}\mathbb{1} \\
& \text{with} \quad \varphi_{s,t,T,V} = \int_s^t E_{u,T}\sigma_u\rho\sigma_u E_{u,V} du
\end{aligned}
\tag{296}
$$

8.4.2. Swaption

The valuation of swaptions is more sophisticated than that of cap/floors due to the fact that the summation on cashflows appears within the (convex) payoff function. Knowing from (293) that the bond $P_{T,V} = P_{T,T,V}$ in Linear Gaussian model (where σ_t is deterministic) is a lognormal martingale under $\mathbb{Q}^T$ measure

$$\frac{P_{T,V}}{P_{s,T,V}} = \exp\left(-\mathbb{1}'\frac{B_{T,V}\varphi_{s,T,T,T}B_{T,V}}{2}\mathbb{1} - \mathbb{1}'B_{T,V}Z_T\right), \qquad Z_T = \int_s^T E_{u,T}\sigma_u dW_u^T \sim \mathcal{N}\left(0, \varphi_{s,T,T,T}\right) \tag{297}$$

where Z_T is n-dimensional normal random variable. Based on (245), the payer swaption price formula (the formula for a receiver swaption can be derived in a similar manner) can be expressed as follows

$$\begin{aligned} V_{s,T,a,b}^{\text{PS}} &= \int_{\mathbb{R}^n}\left(\sum_{i=a}^{b} c_i P_{s,i}\exp\left(-\mathbb{1}'\frac{B_{T,i}\varphi_{s,T,T,T}B_{T,i}}{2}\mathbb{1} - \mathbb{1}'B_{T,i}z\right)\right)^+ f(z)dz \\ &= \int_{\mathbb{R}^n}\left(\sum_{i=a}^{b}\delta_i \exp\left(-\mathbb{1}'B_{T,i}z\right)\right)^+ f(z)dz, \qquad \delta_i = c_i P_{s,i}\exp\left(-\mathbb{1}'\frac{B_{T,i}\varphi_{s,T,T,T}B_{T,i}}{2}\mathbb{1}\right) \end{aligned} \tag{298}$$

where the function $f(z)\colon \mathbb{R}^n \longmapsto \mathbb{R}_+$ is the joint density of n-dimensional normal with mean zero and covariance $\varphi_{s,T,T,T}$

$$f(z) = (2\pi)^{-\frac{n}{2}}\left|\varphi_{s,T,T,T}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}z'\varphi_{s,T,T,T}^{-1}z\right) \tag{299}$$

In general, the multi-dimensional integral must be calculated numerically, however under certain circumstances, analytical or semi-analytical solutions/approximations can be sought. In the following, we will introduce a few methods that are frequently used in practice.

8.4.2.1. *One-Factor Model: Jamshidian's Decomposition*

When the model has only one stochastic factor, all the matrices degenerate into scalars. Assuming that swaption expiry $T = T_a$ and the swap strike rate K is positive, the payer swaption price in (245) simplifies to

$$V_{s,T,a,b}^{\text{PS}} = P_{s,T}\mathbb{E}_s^T\left[\left(1 - \sum_{i=a+1}^{b}|c_i|P_{T,i}\right)^+\right] \tag{300}$$

with the forward bond price

$$P_{T,i} = P_{s,i}\exp\left(-\frac{1}{2}\xi_i^2 - \xi_i Z\right), \qquad \xi_i^2 = \xi_{s,T,T,i}^2 = B_{T,i}^2\varphi_{s,T,T,T} \tag{301}$$

where Z is a standard normal random variable. Since the bond price $P_{T,i}$ admits an affine term structure, we can find a solution z^* such that the fixed leg is valued at par

$$\sum_{i=a+1}^{b} |c_i| P_{T,i}^* = 1 \quad \text{with} \quad P_{T,i}^* = P_{s,T,i} \exp\left(-\frac{1}{2}\xi_i^2 - \xi_i z^*\right), \tag{302}$$

Hence the (300) can be rewritten into

$$V_{s,T,a,b}^{\mathrm{PS}} = P_{s,T}\mathbb{E}_s^T\left[\left(\sum_{i=a+1}^{b}|c_i|P_{T,i}^* - \sum_{i=a+1}^{b}|c_i|P_{T,i}\right)^+\right] = P_{s,T}\mathbb{E}_s^T\left[\left(\sum_{i=a+1}^{b}|c_i|\left(P_{T,i}^* - P_{T,i}\right)\right)^+\right] \tag{303}$$

Since volatility ξ_i is non-negative, the $P_{T,i}$ is a monotonically decreasing function of z (i.e., $P_{T,i}^* - P_{T,i} > 0\ \forall\ Z > z^*$). Hence the call option on a portfolio in (303) can be decomposed into a portfolio of call options with strikes $P_{T,i}^*$ [37], that is

$$\begin{aligned} V_{s,T,a,b}^{\mathrm{PS}} &= P_{s,T}\mathbb{E}_s^T\left[\sum_{i=a+1}^{b}|c_i|\left(P_{T,i}^* - P_{T,i}\right)^+\right] = P_{s,T}\mathbb{E}_s^T\left[\mathbb{1}\{Z > z^*\}\sum_{i=a+1}^{b}|c_i|\left(P_{T,i}^* - P_{T,i}\right)\right] \\ &= P_{s,T}\mathbb{E}_s^T\left[\sum_{i=a+1}^{b}|c_i|\left(P_{T,i}^* - P_{T,i}\right)\mathbb{1}\{Z > z^*\}\right] = P_{s,T}\sum_{i=a+1}^{b}|c_i|\mathbb{E}_s^T\left[\left(P_{T,i}^* - P_{T,i}\right)^+\right] \end{aligned} \tag{304}$$

Since $P_{T,i}$ under $\mathbb{Q}^T$ is a lognormal martingale, the swaption value becomes

$$V_{s,T,a,b}^{\mathrm{PS}} = P_{s,T}\sum_{i=a+1}^{b} c_i \mathfrak{B}\left(P_{T,i}^*, P_{s,T,i}, \xi_i^2, 1\right) \tag{305}$$

where $\mathfrak{B}(\cdot)$ is the Black formula in (81). Note that the formula (305) is developed under the assumption that the swap rate strike K is positive. This assumption however will not hold for certain sovereign interest rates after 2008 financial crisis, which renders the formula invalid. This issue can be addressed by another method, which will be discussed next.

8.4.2.2. *One-Factor Model: Henrard's Method*

There is another method proposed by Henrard [38] in 2003 for swaption valuation. It is basically a variant of the Jamshidian's decomposition. Let us start from the payer swaption price in (298), in one-factor model it reduces to

$$V_{s,T,a,b}^{\mathrm{PS}} = \int_{\mathbb{R}^n} \left(\sum_{i=a}^{b} \delta_i \exp(-\xi_i z) \right)^+ \phi(z) dz\,, \qquad \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2\right) \tag{306}$$

where $\xi_i = \xi_{s,T,T,i}$ for brevity. Let $h(z)$ be the payer swap payoff function in (306), that is

$$h(z) = \sum_{i=a}^{b} \delta_i \exp(-\xi_i z) \qquad \text{with} \qquad \delta_i = c_i P_{s,i} \exp\left(-\frac{1}{2} \xi_i^2\right) \tag{307}$$

The $h(z)$ can be regarded as a sum of exponentially decayed δ_i with non-negative decaying factor ξ_i. Since δ_i has the same sign of c_i, we can imagine that the δ_i's are all positive up to a certain $i = k$ (e.g., $i = a$ for a positive K or $i = b - 1$ for a negative K (given that the K is not too negative such that the last c_b can still be negative)), then all negative. Let us define another axillary function

$$g(z) = h(z) \exp(\xi_k z) = \sum_{i=a}^{b} \delta_i \exp\big((\xi_k - \xi_i) z\big) \tag{308}$$

Because ξ_i is monotonically increasing as bond maturity grows (i.e., $\xi_i < \xi_{i+1}$ for $T_{i+1} > T_i$), the δ_i and the $(\xi_k - \xi_i)$ now have the same sign. Therefore $g(z)$ is strictly increasing. Since $g(z)$ is negative when $z \to -\infty$ and positive when $z \to +\infty$, the monotonicity in $g(z)$ ensures that there is one and only one solution z^* such that $g(z^*) = 0$, and so is it for $h(z)$. In other words, given the unique z^*, the $h(z) < 0$ if $z < z^*$ and $h(z) \geq 0$ otherwise. Hence the payer swaption price in (306) can be transformed into

$$\begin{aligned} V_{s,T,a,b}^{\mathrm{PS}} &= \int_{z^*}^{\infty} \sum_{i=a}^{b} \delta_i \exp(-\xi_i z)\, \phi(z) dz = \sum_{i=a}^{b} \delta_i \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - \xi_i z\right) dz \\ &= \sum_{i=a}^{b} \delta_i \exp\left(\frac{1}{2} \xi_i^2\right) \big(1 - \Phi(z^* + \xi_i)\big) = \sum_{i=a}^{b} c_i P_{s,i} \Phi(-z^* - \xi_i) \end{aligned} \tag{309}$$

using the identity

$$\int_a^b \exp\left(-\frac{\alpha}{2} x^2 - \beta x\right) dx = \frac{\sqrt{2\pi}}{\sqrt{\alpha}} \exp\left(\frac{\beta^2}{2\alpha}\right) \left(\Phi\left(b\sqrt{\alpha} + \frac{\beta}{\sqrt{\alpha}}\right) - \Phi\left(a\sqrt{\alpha} + \frac{\beta}{\sqrt{\alpha}}\right)\right) \;\; \forall \;\; \alpha > 0 \tag{310}$$

where $\Phi(\cdot)$ is the standard normal cumulative density function. In the case of a receiver swaption, the signs of c_i's (and thus the signs of δ_i's) are flipped. The same argument still applies, which gives the receiver swaption price as

$$V_{s,T,a,b}^{\mathrm{RS}} = -\sum_{i=a}^{b} \delta_i \int_{-\infty}^{z^*} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2} - \xi_i z\right) dz = -\sum_{i=a}^{b} c_i P_{s,i} \Phi(z^* + \xi_i) \tag{311}$$

This is consistent with the put-call parity in swaptions, that is, the underlying swap value should be the payer swaption premium minus the receiver swaption premium.

Note that the formulas (309) and (311) are applicable only if the solution z^* is unique. The argument that δ_i's are all positive (negative) up to a certain $i = k$ then all negative (positive) is a sufficient but unnecessary condition for the uniqueness of z^*. It ensures δ_i and $(\xi_k - \xi_i)$ having the same sign and therefore the monotonicity in $g(z)$. This condition looks natural in the single curve framework as coupons are positive, but it may not hold in multi-curve framework (see section 8.5). However, even if the condition was not satisfied (i.e., the δ_i's change several times of sign, so do the c_i's) and the monotonicity in $g(z)$ could not be guaranteed, there would still be a good chance to have a unique z^*, especially when the sizes of irregular δ_i's are reasonably small [39]. Nevertheless, if the z^* is however not unique, the exercise domain of an option will be a union of disjoint intervals rather than a single interval, calculation of the integral must then be done by numerical integration methods.

8.4.2.3. *Two-Factor Model: Numerical Integration*

In 2-factor model, it becomes a bit more sophisticated. Again, we start from the payer swaption price in (298) and convert it explicitly into two factors z_1 and z_2 with joint density $f(z_1, z_2)$

$$V_{s,T,a,b}^{\mathrm{PS}} = \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{i=a}^{b} \delta_i \exp\left(-\xi_{1;i} z_1 - \xi_{2;i} z_2\right) \right)^+ f(z_1, z_2) dz_2 \, dz_1$$
$$\text{and} \quad f(z_1, z_2) = \frac{1}{2\pi} \exp\left(-\frac{z_1^2 + z_2^2}{2}\right) \tag{312}$$

where we define

$$\mathcal{D} = \left|\varphi_{s,T,T,T}\right| = \varphi_{11}\varphi_{22} - \varphi_{12}^2, \qquad \varphi_{ij} = \left[\varphi_{s,T,T,T}\right]_{i,j} = \int_s^T E_{i;u,T}\sigma_{i;u}\rho_{ij}\sigma_{j;u}E_{j;u,T}du \tag{313}$$

The double integral in (312) can be reduced into a single integral by using the aforementioned Jamshidian's trick, that is, for any given value of z_1, we can find a unique solution z_2^* such that the swap payoff $h(z_2)$ equals zero

$$h(z_2^*) = \sum_{i=a}^{b} \delta_i \exp\left(-B_{1;T,i}z_1 - B_{2;T,i}z_2^*\right) = 0 \tag{314}$$

By the same argument, we can show that $h(z_2) < 0$ if $z_2 < z_2^*$ else $h(z_2) \geq 0$. As such, the payer swaption price becomes

$$\begin{aligned} V_{s,T,a,b}^{\mathrm{PS}} &= \int_{\mathbb{R}} \int_{z_2^*}^{\infty} \sum_{i=a}^{b} \delta_i \exp\left(-B_{1;T,i}z_1 - B_{2;T,i}z_2\right) f(z_1, z_2) dz_2\, dz_1 \\ &= \int_{\mathbb{R}} \sum_{i=a}^{b} \delta_i \exp\left(-B_{1;T,i}z_1\right) \int_{z_2^*}^{\infty} \exp\left(-B_{2;T,i}z_2\right) f(z_1, z_2) dz_2\, dz_1 \end{aligned} \tag{315}$$

We can calculate the inner integral in (315) using (310) as

$$\begin{aligned} &\int_{z_2^*}^{\infty} \exp\left(-B_{2;T,i}z_2\right) f(z_1, z_2) dz_2 \\ &= \int_{z_2^*}^{\infty} \frac{1}{2\pi\sqrt{\mathcal{D}}} \exp\left(-B_{2;T,i}z_2 - \frac{\varphi_{22}z_1^2 - 2\varphi_{12}z_1z_2 + \varphi_{11}z_2^2}{2\mathcal{D}}\right) dz_2 \\ &= \frac{1}{2\pi\sqrt{\mathcal{D}}} \exp\left(-\frac{\varphi_{22}z_1^2}{2\mathcal{D}}\right) \int_{z_2^*}^{\infty} \exp\left(-\frac{\varphi_{11}z_2^2}{2\mathcal{D}} - \left(B_{2;T,i} - \frac{\varphi_{12}z_1}{\mathcal{D}}\right) z_2\right) dz_2 \\ &= \frac{1}{\sqrt{2\pi\varphi_{11}}} \exp\left(\frac{\mathcal{D}}{2\varphi_{11}}\left(B_{2;T,i} - \frac{\varphi_{12}z_1}{\mathcal{D}}\right)^2 - \frac{\varphi_{22}z_1^2}{2\mathcal{D}}\right)\left(1 - \Phi\left(z_2^*\sqrt{\frac{\varphi_{11}}{\mathcal{D}}} - \frac{\varphi_{12}z_1 - B_{2;T,i}\mathcal{D}}{\sqrt{\varphi_{11}\mathcal{D}}}\right)\right) \\ &= \frac{1}{\sqrt{2\pi\varphi_{11}}} \exp\left(\frac{\mathcal{D}B_{2;T,i}^2}{2\varphi_{11}} - \frac{\varphi_{12}B_{2;T,i}z_1}{\varphi_{11}} + \frac{\varphi_{12}^2z_1^2}{2\varphi_{11}\mathcal{D}} - \frac{\varphi_{22}z_1^2}{2\mathcal{D}}\right) \Phi\left(\frac{\varphi_{12}z_1 - B_{2;T,i}\mathcal{D}}{\sqrt{\varphi_{11}\mathcal{D}}} - z_2^*\sqrt{\frac{\varphi_{11}}{\mathcal{D}}}\right) \\ &= \frac{1}{\sqrt{2\pi\varphi_{11}}} \exp\left(\frac{\mathcal{D}B_{2;T,i}^2 - 2\varphi_{12}B_{2;T,i}z_1 - z_1^2}{2\varphi_{11}}\right) \Phi\left(\frac{\varphi_{12}z_1 - B_{2;T,i}\mathcal{D}}{\sqrt{\varphi_{11}\mathcal{D}}} - z_2^*\sqrt{\frac{\varphi_{11}}{\mathcal{D}}}\right) \end{aligned} \tag{316}$$

Using (316) to substitute for the inner integral in (315), we find

$$
\begin{aligned}
V_{s,T,a,b}^{\mathrm{PS}} &= \int_{\mathbb{R}} \sum_{i=a}^{b} \delta_i \exp\left(-B_{1;T,i} z_1\right) \int_{z_2^*}^{\infty} \exp\left(-B_{2;T,i} z_2\right) f(z_1, z_2) dz_2 \, dz_1 \\
&= \int_{\mathbb{R}} \sum_{i=a}^{b} \frac{\delta_i \exp\left(\frac{\mathcal{D} B_{2;T,i}^2 - 2\varphi_{12} B_{2;T,i} z_1 - z_1^2}{2\varphi_{11}} - B_{1;T,i} z_1\right) \Phi\left(\frac{\varphi_{12} z_1 - B_{2;T,i}\mathcal{D}}{\sqrt{\varphi_{11}\mathcal{D}}} - z_2^* \sqrt{\frac{\varphi_{11}}{\mathcal{D}}}\right)}{\sqrt{2\pi\varphi_{11}}} dz_1
\end{aligned}
\tag{317}
$$

$$
\text{where} \quad \delta_i = c_i P_{s,i} \exp\left(-\mathbb{1}' \frac{B_{T,i} \varphi_{s,T,T,T} B_{T,i}}{2} \mathbb{1}\right)
$$

This integral can be calculated numerically, for example, by Gauss-Hermite quadrature[1], to form a semi-analytical formula.

8.4.2.4. *Multi-Factor Model: Swap Rate Approximation*

It is awkward to calculate the integrals numerically when efficiency is highly demanded. Instead, we may seek an approximative but sufficiently accurate solution for the swaption price. Knowing that a swaption is actually a contingent claim on swap rate, we may price the payer swaption using (88)

$$
V_{s,T,a,b}^{\mathrm{PS}} = A_s^{a,b} \mathbb{E}_s^{a,b}\left[\left(S_T^{a,b} - K\right)^+\right], \qquad S_t^{a,b} = \frac{P_{t,a} - P_{t,b}}{A_t^{a,b}}, \qquad A_t^{a,b} = \sum_{i=a+1}^{b} \tau_i P_{t,i} \tag{318}
$$

The swap rate $S_t^{a,b}$ for $s < t < T$ is a martingale under the swap measure $\mathbb{Q}^{a,b}$ with annuity $A_t^{a,b}$ as the numeraire. We may express its dynamics in terms of the stochastic factor x_t in (289), that is

$$
dS_t^{a,b} = (\cdot)dt + \frac{\partial S_t^{a,b}}{\partial x} \sigma_t d\widetilde{W}_t = J_t \sigma_t dW_t^{a,b} \tag{319}
$$

where J_t is the Jacobian (a row vector) of $S_t^{a,b}$ with respect to factor x_t. Based on the bond price (292), the k-th element of J_t reads

[1]A 12-point Gauss-Hermite quadrature would be able to provide sufficient accuracy.
.

$$\begin{aligned}[J_t]_k = \frac{\partial S_t^{a,b}}{\partial x_k} &= -\frac{P_{t,a}B_{k;t,a}}{A_t^{a,b}} + \frac{P_{t,b}B_{k;t,b}}{A_t^{a,b}} + \frac{P_{t,a} - P_{t,b}}{A_t^{a,b}} \frac{\sum_{i=a+1}^{b} \tau_i P_{t,i} B_{k;t,i}}{A_t^{a,b}} \\ &= \frac{P_{t,b}}{A_t^{a,b}} B_{k;t,b} - \frac{P_{t,a}}{A_t^{a,b}} B_{k;t,a} + \frac{S_t^{a,b}}{A_t^{a,b}} \sum_{i=a+1}^{b} \tau_i P_{t,i} B_{k;t,i}\end{aligned} \tag{320}$$

and hence

$$J_t = \mathbb{1}' \sum_{i=a}^{b} \eta_{t,i} B_{t,i} \quad \text{for} \quad \eta_{t,i} = \begin{cases} -\dfrac{P_{t,a}}{A_t^{a,b}} & \text{if } \ i = a \\ \dfrac{S_t^{a,b} \tau_i P_{t,i}}{A_t^{a,b}} & \text{if } \ a+1 \le i \le b-1 \\ \dfrac{\left(1 + S_t^{a,b} \tau_b\right) P_{t,b}}{A_t^{a,b}} & \text{if } \ i = b \end{cases} \tag{321}$$

Fixing the stochastic term $P_{t,i}$ and $A_t^{a,b}$ in $\eta_{t,i}$ with their values at start time s (i.e., using the trick of "freezing the initial values"), we can have an approximative but deterministic Jacobian vector J_s such that

$$dS_t^{a,b} \approx J_s \sigma_t dW_t^{a,b} \quad \text{for} \quad J_s = \mathbb{1}' \sum_{i=a}^{b} \eta_{s,i} B_{t,i} \tag{322}$$

This approximation shows that the martingale $S_t^{a,b}$ is (approximately) normally distributed under $\mathbb{Q}^{a,b}$ measure and its total variance, denoted by v, can be calculated by

$$v = \int_s^T J_s \sigma_t \rho \sigma_t J_s' dt = \int_s^T \mathbb{1}' \sum_{i=a}^{b} \eta_{s,i} B_{t,i} \, \sigma_t \rho \sigma_t \sum_{i=a}^{b} \eta_{s,i} B_{t,i} \, \mathbb{1}' dt = \sum_{i,j=a}^{b} \eta_{s,i} \eta_{s,j} \mathbb{1}' \psi_{s,T,i,j} \mathbb{1} \tag{323}$$

with $\psi_{s,T,i,j}$ defined in (229) for $b_{t,T} = B_{t,T}\sigma_t$. The payer swaption can then be priced using *Bachelier* formula

$$\begin{aligned}V_{s,T,a,b}^{\mathrm{PS}} &= A_s^{a,b} \mathbb{E}_s^{a,b} \left[\left(S_T^{a,b} - K\right)^+ \right] = A_s^{a,b} \mathbb{E}_s^{a,b} \left[\left(S_s^{a,b} + Z\sqrt{v} - K\right)^+ \right] \\ &= A_s^{a,b} \int_{\mathbb{R}} \left(S_s^{a,b} + z\sqrt{v} - K\right)^+ \phi(z) dz = A_s^{a,b} \left(\left(S_s^{a,b} - K\right) \Phi\left(\frac{S_s^{a,b} - K}{\sqrt{v}}\right) + \sqrt{v}\,\phi\left(\frac{S_s^{a,b} - K}{\sqrt{v}}\right) \right)\end{aligned} \tag{324}$$

where Z is a standard normal random variable. Similarly, we can derive the receiver swaption formula as

$$V_{s,T,a,b}^{\mathrm{RS}} = A_s^{a,b}\mathbb{E}_s^{a,b}\left[\left(K - S_T^{a,b}\right)^+\right] = A_s^{a,b}\left(\left(K - S_s^{a,b}\right)\Phi\left(\frac{K - S_s^{a,b}}{\sqrt{v}}\right) + \sqrt{v}\phi\left(\frac{K - S_s^{a,b}}{\sqrt{v}}\right)\right) \tag{325}$$

8.4.3. CMS Spread Option Approximation

A CMS spread caplet[1], with a payment usually occurs at T_p, is typically valued by

$$V_{s,T}^{\mathrm{CMSSC}} = P_{s,T}\mathbb{E}_s^T\left[P_{T,p}\left(S_T^{a,b} - S_T^{c,d} - K\right)^+\right] = P_{s,p}\mathbb{E}_s^p\left[\left(S_T^{a,b} - S_T^{c,d} - K\right)^+\right] \tag{326}$$

for $s < t < T \le T_a = T_c < T_p < T_b < T_d$. The $S_t^{a,b}$ and $S_t^{c,d}$ are two CMS rates fixed at T with different tenors (e.g., 2Y and 10Y). In order to compute the expectation, we need to find the joint distribution of the swap rates under T_p-forward measure.

Given (292), we can write the dynamics of the bond $P_{t,p}$ and then the annuity $A_t^{a,b}$ in (318) as

$$\begin{aligned} dP_{t,p} &= dP_{t,p}\left(r_t dt - \mathbb{1}' B_{t,p}\sigma_t d\widetilde{W}_t\right) \\ dA_t^{a,b} &= \sum_{i=a+1}^{b}\tau_i P_{t,i}\left(r_t dt - \mathbb{1}' B_{t,i}\sigma_t d\widetilde{W}_t\right) = A_t^{a,b}\left(r_t dt - \mathbb{1}'\frac{1}{A_t^{a,b}}\sum_{i=a+1}^{b}\tau_i P_{t,i}B_{t,i}\sigma_t\, d\widetilde{W}_t\right) \end{aligned} \tag{327}$$

Using formula (30), we can change from the swap measure $\mathbb{Q}^{a,b}$ with annuity $A_t^{a,b}$ as the numeraire to the T_p-forward measure with bond $P_{t,p}$ as the numeraire

$$\begin{aligned} dW_t^{a,b} &= dW_t^p + \rho\left(-\sigma_t B_{t,p} + \frac{\sigma_t}{A_t^{a,b}}\sum_{i=a+1}^{b}\tau_i P_{t,i}B_{t,i}\right)\mathbb{1}dt = dW_t^p + \rho\sigma_t\sum_{i=p,a+1}^{b} w_{t,i}B_{t,i}\,\mathbb{1}dt \\ \text{for}\quad w_{t,i} &= \begin{cases} -1 & \text{if}\quad i = p \\ \dfrac{\tau_i P_{t,i}}{A_t^{a,b}} & \text{if}\quad a+1 \le i \le b \end{cases} \end{aligned} \tag{328}$$

An approximation can be made by freezing $w_{t,i}$ at start time s. And then substituting (328) into the swap rate dynamics (322), we find that

[1] Here we talk about CMS spread option we mean a CMS spread caplet/floorlet.
.

$$
\begin{aligned}
dS_t^{a,b} &\approx J_s\sigma_t dW_t^p + \mathbb{1}'\sum_{i=a}^{b}\eta_{s,i}B_{t,i}\,\sigma_t\rho\sigma_t\sum_{j=p,a+1}^{b} w_{s,j}B_{t,j}\,\mathbb{1}dt \\
&= J_s\sigma_t dW_t^p + \sum_{i=a}^{b}\sum_{j=p,a+1}^{b}\eta_{s,i}w_{s,j}\mathbb{1}'B_{t,i}\sigma_t\rho\sigma_t B_{t,j}\mathbb{1}\,dt
\end{aligned}
\tag{329}
$$

The swap rate is (approximately) normally distributed under T_p-forward measure with mean and variance as

$$
\mathbb{E}_s^p\left[S_T^{a,b}\right] = S_s^{a,b} + \sum_{i=a}^{b}\sum_{j=p,a+1}^{b}\eta_{s,i}w_{s,j}\mathbb{1}'\psi_{s,T,i,j}\mathbb{1}\,,\qquad \mathbb{V}_s^p\left[S_T^{a,b}\right] = \sum_{i,j=a}^{b}\eta_{s,i}\eta_{s,j}\mathbb{1}'\psi_{s,T,i,j}\mathbb{1}
\tag{330}
$$

If we denote $\delta_t = S_t^{a,b} - S_t^{c,d}$the spread between the two swap rate, because each swap rate is normal, the spread is also (approximately) normal with mean and variance as follows

$$
\begin{aligned}
\mu &= \mathbb{E}_s^p[\delta_T] = S_s^{a,b} - S_s^{c,d} + \sum_{i=a}^{b}\sum_{j=p,a+1}^{b}\eta_{s,i}w_{s,j}\mathbb{1}'\psi_{s,T,i,j}\mathbb{1} - \sum_{i=c}^{d}\sum_{j=p,c+1}^{d}\eta_{s,i}w_{s,j}\mathbb{1}'\psi_{s,T,i,j}\mathbb{1} \\
v &= \mathbb{V}_s^p[\delta_T] = \sum_{i,j=a}^{b}\eta_{s,i}\eta_{s,j}\mathbb{1}'\psi_{s,T,i,j}\mathbb{1} + \sum_{k,l=c}^{d}\eta_{s,k}\eta_{s,l}\mathbb{1}'\psi_{s,T,k,l}\mathbb{1} - 2\sum_{i=a}^{b}\sum_{k=c}^{d}\eta_{s,i}\eta_{s,k}\mathbb{1}'\psi_{s,T,i,k}\mathbb{1}
\end{aligned}
\tag{331}
$$

Hence the CMS spread caplet and floorlet can be priced using *Bachelier's formula* as

$$
\begin{aligned}
V_{s,T}^{\text{CMSSC}} &= P_{s,p}\mathbb{E}_s^p[(\delta_T - K)^+] = P_{s,p}\mathbb{E}_s^p\left[\left(\mu + Z\sqrt{v} - K\right)^+\right] = P_{s,p}\int_{\mathbb{R}}\left(\mu + z\sqrt{v} - K\right)^+\phi(z)dz \\
&= P_{s,p}\left((\mu - K)\Phi\left(\frac{\mu - K}{\sqrt{v}}\right) + \sqrt{v}\phi\left(\frac{\mu - K}{\sqrt{v}}\right)\right) \\
V_{s,T}^{\text{CMSSF}} &= P_{s,p}\mathbb{E}_s^p[(K - \delta_T)^+] = P_{s,p}\left((K - \mu)\Phi\left(\frac{K - \mu}{\sqrt{v}}\right) + \sqrt{v}\phi\left(\frac{K - \mu}{\sqrt{v}}\right)\right)
\end{aligned}
\tag{332}
$$

In the case of a digital CMS spread caplet (or floorlet) that pays \$1 at time T_p if $\delta_T - K > 0$ (or $\delta_T - K < 0$ for floorlet), its value at time s can be priced by

$$
V_{s,T}^{\text{CMSDSC}} = P_{s,p}\mathbb{E}_s^p\left[\mathbb{1}_{\{\delta_T - K > 0\}}\right] = P_{s,p}\Phi\left(\frac{\mu - K}{\sqrt{v}}\right)
\tag{333}
$$

$$V_{s,T}^{\text{CMSDSF}} = P_{s,p}\mathbb{E}_s^p\left[\mathbb{1}_{\{K-\delta_T>0\}}\right] = P_{s,p}\Phi\left(\frac{K-\mu}{\sqrt{v}}\right)$$

8.5. One-Factor Hull-White Model in Multi-Curve Framework

In practical applications, the short rate model in (295) usually takes a time-invariant κ along with a (deterministic) piecewise constant σ, such that $\sigma_t = \sigma_i \ \forall\, t_{i-1} < t \le t_i$, which effectively defines a Linear Gaussian model (a.k.a. Hull-White model). The reason a time variant κ is *not* in favor is that it makes the evolution of forward rate volatility strongly non-stationary. This has been intensively discussed in [40]. In the case of single factor model, the short rate and its driving process further simplifies into

$$\begin{aligned} & r_t = f_{s,t} + x_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \varphi_{s,t,t,t} - \kappa x_t\right)dt + \sigma_t d\widetilde{W}_t \quad \text{with} \\ & x_t = \chi_{s,t,t,t} + \int_s^t E_{u,t}\sigma_u d\widetilde{W}_u\,, \qquad dx_t = \left(\varphi_{s,t,t,t} - \kappa x_t\right)dt + \sigma_t d\widetilde{W}_t \end{aligned} \tag{334}$$

where by a constant κ we have

$$\begin{aligned} & E_{t,T} = e^{-\kappa(T-t)}, \qquad \lim_{\kappa\to 0} E_{t,T} = 1 \qquad \text{and} \\ & B_{t,T} = \int_t^T E_{t,u}du = \frac{1-e^{-\kappa(T-t)}}{\kappa}, \qquad \lim_{\kappa\to 0} B_{t,T} = T-t \end{aligned} \tag{335}$$

With the assumption of the piecewise constant volatility σ_t, the auxiliary variance/covariance terms defined in (229) can be further specialized into

$$\begin{aligned} & \varphi_{s,t,T,V} = \int_s^t E_{u,T}E_{u,V}\sigma_u^2 du = \sum_s^t \sigma_i^2 e^{-\kappa(T+V)}\gamma_{i;2} \\ & \chi_{s,t,T,V} = \int_s^t E_{u,T}B_{u,V}\sigma_u^2 du = \sum_s^t \sigma_i^2 \frac{e^{-\kappa T}\gamma_{i;1} - e^{-\kappa(T+V)}\gamma_{i;2}}{\kappa} \\ & \psi_{s,t,T,V} = \int_s^t B_{u,T}B_{u,V}\sigma_u^2 du = \sum_s^t \sigma_i^2 \frac{\gamma_{i;0} - (e^{-\kappa T} + e^{-\kappa V})\gamma_{i;1} + e^{-\kappa(T+V)}\gamma_{i;2}}{\kappa^2} \end{aligned} \tag{336}$$

$$\text{where} \qquad \gamma_{i;n} = \int_{t_{i-1}}^{t_i} e^{n\kappa u}du = \frac{e^{n\kappa t_i} - e^{n\kappa t_{i-1}}}{n\kappa} \quad \text{for} \quad n = 0, 1, 2$$

When $\kappa \to 0$, they reduce to

$$\lim_{\kappa\to 0}\varphi_{s,t,T,V} = \int_s^t \sigma_u^2 du = \sum_s^t \sigma_i^2 \delta_{i;0}$$

$$\lim_{\kappa\to 0}\chi_{s,t,T,V} = \int_s^t (V-u)\sigma_u^2 du = \sum_s^t \sigma_i^2 \left(V\delta_{i;0} - \delta_{i;1}\right) \tag{337}$$

$$\lim_{\kappa\to 0}\psi_{s,t,T,V} = \int_s^t (T-u)(V-u)\sigma_u^2 du = \sum_s^t \sigma_i^2 \left(TV\delta_{i;0} - (T+V)\delta_{i;1} + \delta_{i;2}\right)$$

$$\text{where} \qquad \delta_{i;n} = \int_{t_{i-1}}^{t_i} u^n du = \frac{t_i^{n+1} - t_{i-1}^{n+1}}{n+1} \quad \text{for} \quad n = 0, 1, 2$$

Note that we always have $\gamma_{i;0} = \delta_{i;0} = t_i - t_{i-1}$.

In the following, we will discuss the model and its calibration as well as its numerical methods in product pricing. A good reference that covers this topic can be found in [41].

8.5.1. Zero Coupon Bond

Let us write the total variance of a forward bond $P_{t,T,V}$ as in (296) by

$$\xi_{s,t,T,V}^2 = B_{T,V}^2 \int_s^t E_{u,T}^2 \sigma_u^2 du = B_{T,V}^2 \varphi_{s,t,T,T} \tag{338}$$

The forward bond price (293) becomes

$$\begin{aligned}
\frac{P_{t,T,V}}{P_{s,T,V}} &= \exp\left(-\frac{1}{2}\int_s^t \left(B_{u,V}^2 - B_{u,T}^2\right)\sigma_u^2 du - B_{T,V}\int_s^t E_{u,T}\sigma_u d\widetilde{W}_u\right) \\
&= \exp\left(-\frac{\psi_{s,t,V,V} - \psi_{s,t,T,T}}{2} - \xi_{s,t,T,V}\tilde{Z}_{s,t}\right) \\
&= \exp\left(-\frac{1}{2}B_{T,V}^2 \int_s^t E_{u,T}^2 \sigma_u^2 du - B_{T,V}\int_s^t E_{u,T}\sigma_u dW_u^T\right) \\
&= \exp\left(-\frac{1}{2}\xi_{s,t,T,V}^2 - \xi_{s,t,T,V} Z_{s,t}^T\right)
\end{aligned} \tag{339}$$

$$\psi_{s,t,V,V} - \psi_{s,t,T,T} = B_{T,V}^2\, \psi_{s,t,T,T} + 2B_{T,V}\chi_{s,t,T,T}$$

where $\tilde{Z}_{s,t}$ and $Z_{s,t}^{T}$ are standard normal random variables under risk neutral measure $\mathbb{Q}$ and T-forward measure $\mathbb{Q}^T$ respectively, and are independent of $\mathcal{F}_s$. The (spot) bond price $P_{t,T}$, which is equivalent to $P_{t,t,T}$, reads

$$P_{t,T} = \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{\psi_{s,t,T,T} - \psi_{s,t,t,t}}{2} - \xi_{s,t,t,T}\tilde{Z}_{s,t}\right) \tag{340}$$

8.5.2. Constant Spread Assumption

To simplify the modeling, we may assume constant spread between the projection and the discount curve. In the following, two types of assumptions are discussed. The first to be considered is the constant additive spread, which assumes a time-invariant spread between the i-th Libor rate $\hat{L}_{t,i}$ and the rate $L_{t,i}$, that is

$$\delta_i = \hat{L}_{t,i} - L_{t,i} = \frac{1}{c_{i,f}}\left(\frac{\hat{P}(t, f_{i,s})}{\hat{P}(t, f_{i,e})} - \frac{P(t, f_{i,s})}{P(t, f_{i,e})}\right) = \frac{1}{c_{i,f}}\left(\frac{\hat{P}(s, f_{i,s})}{\hat{P}(s, f_{i,e})} - \frac{P(s, f_{i,s})}{P(s, f_{i,e})}\right) \tag{341}$$

where $L_{t,i}$ is the counterpart of $\hat{L}_{t,i}$ estimated from the discounting curve

$$L_{t,i} = \frac{P(t, f_{i,s}) - P(t, f_{i,e})}{c_{i,f} P(t, f_{i,e})} \tag{342}$$

The second is constant multiplicative spread, which assumes that the quantity

$$\eta_i = \frac{P(t, f_{i,s}, f_{i,e})}{\hat{P}(t, f_{i,s}, f_{i,e})} = \frac{P(s, f_{i,s}, f_{i,e})}{\hat{P}(s, f_{i,s}, f_{i,e})} = \frac{P(s, f_{i,e})}{P(s, f_{i,s})}\frac{\hat{P}(s, f_{i,s})}{\hat{P}(s, f_{i,e})} \tag{343}$$

is time invariant for the i-th Libor rate $\hat{L}_{t,i}$. And hence we have

$$\hat{L}_{t,i} = \eta_i L_{t,i} + \frac{\eta_i - 1}{c_{i,f}} \tag{344}$$

This is indeed equivalent to assuming constant additive spread between continuous compounded zero rates of the projection and discount curve. In general, we would expect $\delta_i > 0$ and $\eta_i > 1$ to account for credit and liquidity spread between the two curves.

Note that under either assumption, the $\hat{L}_{t,i}$ can be expressed as an affine function of the $L_{t,i}$

$$\hat{L}_{t,i} = \alpha_i L_{t,i} + \beta_i \tag{345}$$

with $\alpha_i = 1$, $\beta_i = \delta_i$ in constant additive spread assumption and $\alpha_i = \eta_i$, $\beta_i = \frac{\eta_i - 1}{c_{i,f}}$ in constant multiplicative spread assumption. Since the $L_{t,i}$ is a martingale under $f_{i,e}$-forward measure, so is the $\hat{L}_{t,i}$.

Assuming constant spread implies that the dynamics of projection curve and discount curve both are driven by a common stochastic factor, e.g., $\tilde{Z}_{s,t}$ in (340). As for simplicity, it is often in favor of the constant multiplicative spread (343), under which the projection curve and the discounting curve share similar rate dynamics, such as

$$\begin{aligned}\frac{\hat{P}(t,T)}{\hat{P}(s,t,T)} = \frac{P(t,T)}{P(s,t,T)} &= \exp\left(-\frac{\psi_{s,t,T,T} - \psi_{s,t,t,t}}{2} - \xi_{s,t,t,T}\tilde{Z}_{s,t}\right) \\ &= \exp\left(-\frac{\xi_{s,t,t,T}^2}{2} - B_{t,T}\chi_{s,t,t,t} - \xi_{s,t,t,T}\tilde{Z}_{s,t}\right)\end{aligned} \tag{346}$$

8.5.3. Caplet and Floorlet

Following the notation of swap schedule defined in chapter 3, a Libor rate cap with expiry T can be priced at time s as a portfolio of caplets

$$V_{s,1,m}^{\text{CAP}} = \sum_{i=1}^{m} V_{s,i}^{\text{CPL}} = \sum_{i=1}^{m} P(s,p_i)\mathbb{E}_s^{p_i}\left[\left(\hat{L}_{f_i,i} - K\right)^+ c_{i,a}\right] \tag{347}$$

The i-th caplet, under the constant spread assumption, can be valued using (345)

$$\begin{aligned}V_{s,i}^{\text{CPL}} = P(s,p_i)\mathbb{E}_s^{p_i}\left[\left(\hat{L}_{f_i,i} - K\right)^+ c_{i,a}\right] &= \alpha_i c_{i,a} P(s,p_i)\mathbb{E}_s^{p_i}\left[\left(L_{f_i,i} - \frac{K-\beta_i}{\alpha_i}\right)^+\right] \\ &= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,p_i)\mathbb{E}_s^{p_i}\left[\left(\frac{P(f_i,f_{i,s})}{P(f_i,f_{i,e})} - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}\right)^+\right]\end{aligned} \tag{348}$$

If ignoring the difference between payment date p_i and fixing period end date $f_{i,e}$, the bond price ratio $\frac{P(f_i,f_{i,s})}{P(f_i,f_{i,e})}$ is a lognormal martingale under p_i-forward measure and hence the caplet price (and similarly the floorlet price) can be calculated by *Black* formula (81) as

$$
\begin{aligned}
V_{s,i}^{\text{CPL}} &= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,p_i) \mathfrak{B}\left(\frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}, \frac{P(s,f_{i,s})}{P(s,f_{i,e})}, \xi^2_{s,f_i,f_{i,s},f_{i,e}}, 1\right) \\
V_{s,i}^{\text{FLR}} &= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,p_i) \mathfrak{B}\left(\frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}, \frac{P(s,f_{i,s})}{P(s,f_{i,e})}, \xi^2_{s,f_i,f_{i,s},f_{i,e}}, -1\right)
\end{aligned}
\tag{349}
$$

with $\xi^2_{s,t,T,V}$ defined in (338).

Alternatively, we transform (348) into

$$
\begin{aligned}
V_{s,i}^{\text{CPL}} &= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,p_i) \mathbb{E}_s^{p_i}\left[\left(\frac{P(t,f_{i,s})}{P(t,f_{i,e})} - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}\right)^+\right] \\
&= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,f_i) \mathbb{E}_s^{f_i}\left[P(f_i,p_i)\left(\frac{P(t,f_{i,s})}{P(t,f_{i,e})} - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}\right)^+\right] \\
&= \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,f_i) \mathbb{E}_s^{f_i}\left[\left(P(f_i,f_{i,s}) - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i} P(f_i,f_{i,e})\right)^+\right]
\end{aligned}
\tag{350}
$$

where the last equality holds if assuming the p_i coincides with $f_{i,e}$. Under f_i-forward measure, the bond $P(f_i,T)$ is a lognormal martingale and its price can be derived from (339) such that

$$
P(f_i,T) = P(s,f_i,T)\exp\left(-\frac{1}{2}\xi_T^2 - \xi_T Z^{f_i}\right), \qquad \xi_T^2 = \xi^2_{s,f_i,f_i,T} = B^2_{f_i,T}\varphi_{s,f_i,f_i,f_i}
\tag{351}
$$

Following the argument in section 0, there must be a unique solution z^* for the payoff

$$
\begin{aligned}
&P(f_i,f_{i,s}) - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i} P(f_i,f_{i,e}) = 0 \\
&\Longrightarrow \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i} = \frac{P(f_i,f_{i,s})}{P(f_i,f_{i,e})} = \frac{P(s,f_{i,s})}{P(s,f_{i,e})}\exp\left(\frac{1}{2}(\xi_e^2 - \xi_s^2) + (\xi_e - \xi_s)z^*\right) \\
&\Longrightarrow z^* = \frac{1}{\xi_e - \xi_s}\log\left(\frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i}\frac{P(s,f_{i,e})}{P(s,f_{i,s})}\right) - \frac{\xi_e + \xi_s}{2}
\end{aligned}
\tag{352}
$$

where $\xi_s = \xi_{s,f_i,f_i,f_{i,s}}$ and $\xi_e = \xi_{s,f_i,f_i,f_{i,e}}$. With the z^*, we can calculate the caplet price in (350) by

$$
V_{s,i}^{\text{CPL}} = \alpha_i \frac{c_{i,a}}{c_{i,f}} P(s,f_i) \int_{z^*}^{\infty} \left(P(f_i,f_{i,s}) - \frac{\alpha_i + c_{i,f}(K-\beta_i)}{\alpha_i} P(f_i,f_{i,e})\right)\phi(z)dz
\tag{353}
$$

$$= \alpha_i \frac{c_{i,a}}{c_{i,f}} \left(P(s, f_{i,s}) \int_{z^*}^{\infty} \exp\left(-\frac{\xi_s^2}{2} - \xi_s z\right) \phi(z) dz - \frac{\alpha_i + c_{i,f}(K - \beta_i)}{\alpha_i} P(s, f_{i,e}) \int_{z^*}^{\infty} \exp\left(-\frac{\xi_e^2}{2} - \xi_e z\right) \phi(z) dz \right)$$

$$= \alpha_i \frac{c_{i,a}}{c_{i,f}} \left(P(s, f_{i,s}) \Phi(-z^* - \xi_s) - \frac{\alpha_i + c_{i,f}(K - \beta_i)}{\alpha_i} P(s, f_{i,e}) \Phi(-z^* - \xi_e) \right)$$

In a similar manner, the floorlet price reads

$$V_{s,i}^{\mathrm{FLR}} = \alpha_i \frac{c_{i,a}}{c_{i,f}} \left(-P(s, f_{i,s}) \Phi(z^* + \xi_s) + \frac{\alpha_i + c_{i,f}(K - \beta_i)}{\alpha_i} P(s, f_{i,e}) \Phi(z^* + \xi_e) \right) \tag{354}$$

It can be shown that (353) and (354) are equivalent to (349) by observing that

$$\begin{aligned} &\xi_{s,f_i,f_{i,s},f_{i,e}}^2 = B_{f_{i,s},f_{i,e}}^2 \varphi_{s,f_i,f_{i,s},f_{i,s}} = \left(B_{f_i,f_{i,e}} - B_{f_i,f_{i,s}}\right)^2 \varphi_{s,f_i,f_i,f_i} = (\xi_e - \xi_s)^2 \\ &d^+ = -z^* - \xi_s \qquad \text{and} \qquad d^- = -z^* - \xi_e \end{aligned} \tag{355}$$

When assuming constant multiplicative spread where $\alpha_i = \eta_i$ and $\beta_i = \frac{\eta_i - 1}{c_{i,f}}$, we have the caplet and floorlet price given by (353) and (354) as

$$\begin{aligned} V_{s,i}^{\mathrm{CPL}} &= \frac{c_{i,a}}{c_{i,f}} P(s, p_i) \left(\frac{\hat{P}(s, f_{i,s})}{\hat{P}(s, f_{i,e})} \Phi(-z^* - \xi_s) - \left(1 + K c_{i,f}\right) \Phi(-z^* - \xi_e) \right) \\ V_{s,i}^{\mathrm{FLR}} &= \frac{c_{i,a}}{c_{i,f}} P(s, p_i) \left(-\frac{\hat{P}(s, f_{i,s})}{\hat{P}(s, f_{i,e})} \Phi(z^* + \xi_s) + \left(1 + K c_{i,f}\right) \Phi(z^* + \xi_e) \right) \\ \text{with} \qquad z^* &= \frac{1}{\xi_e - \xi_s} \log\left(\left(1 + K c_{i,f}\right) \frac{\hat{P}(s, f_{i,e})}{\hat{P}(s, f_{i,s})} \right) - \frac{\xi_e + \xi_s}{2} \end{aligned} \tag{356}$$

This is the formula stated in Theorem 1 of [41]. Under the constant additive spread assumption where $\alpha_i = 1$ and $\beta_i = \delta_i$, the caplet and floorlet price are

$$V_{s,i}^{\mathrm{CPL}} = \frac{c_{i,a}}{c_{i,f}} \left(P(s, f_{i,s}) \Phi(-z^* - \xi_s) - \left(1 + c_{i,f}(K - \delta_i)\right) P(s, f_{i,e}) \Phi(-z^* - \xi_e) \right) \tag{357}$$

$$V_{s,i}^{\mathrm{FLR}} = \frac{c_{i,a}}{c_{i,f}}\Big(-P\big(s,f_{i,s}\big)\Phi(z^* + \xi_s) + \Big(1 + c_{i,f}(K-\delta_i)\Big)P\big(s,f_{i,e}\big)\Phi(z^*+\xi_e)\Big)$$

$$\text{with} \qquad z^* = \frac{1}{\xi_e - \xi_s}\log\left(\Big(1 + c_{i,f}(K-\delta_i)\Big)\frac{P\big(s,f_{i,e}\big)}{P\big(s,f_{i,s}\big)}\right) - \frac{\xi_e + \xi_s}{2}$$

Note that the difference between $c_{i,a}$ and $c_{i,f}$ (i.e., fixing and accrual period may differ) is usually negligible, however it must be taken into account when pricing cap/floors in a rigorous setup.

8.5.4. Swaption

We will follow Henrard's method presented in section 8.4.2.2 to derive the swaption formula in multi-curve framework. Following the notations used in Chapter 3, the price of a payer swaption maturing at T is

$$V_{s,T}^{\mathrm{PS}} = P(s,T)\mathbb{E}_s^T\left[\left(\sum_{i=1}^{m}\hat{L}_{T,i}c_{i,a}P(T,p_i) - K\sum_{j=1}^{n}c_{j,a}P\big(T,p_j\big)\right)^+\right] \tag{358}$$

By assuming constant spread $\hat{L}_{t,i} = \alpha_i L_{t,i} + \beta_i$ and $p_i = f_{i,e} = a_{i,e}$ (and hence $c_{i,a} = c_{i,f}$), the floating leg becomes

$$\begin{aligned}
&\sum_{i=1}^{m}\hat{L}_{T,i}c_{i,a}P(T,p_i) = \sum_{i=1}^{m}L_{T,i}c_{i,a}P(T,p_i) + \sum_{i=1}^{m}\big(\hat{L}_{T,i} - L_{T,i}\big)c_{i,a}P(T,p_i)\\
&= \sum_{i=1}^{m}L_{T,i}c_{i,a}P(T,p_i) + \sum_{i=1}^{m}\Big((\alpha_i - 1)L_{T,i} + \beta_i\Big)c_{i,a}P(T,p_i)\\
&= \sum_{i=1}^{m}L_{T,i}c_{i,a}P(T,p_i) + \sum_{i=1}^{m}(\alpha_i - 1)\frac{c_{i,a}}{c_{i,f}}P\big(t,f_{i,s}\big) + \sum_{i=1}^{m}\left(\beta_i - \frac{\alpha_i - 1}{c_{i,f}}\right)c_{i,a}P(T,p_i)\\
&= P(T,t_e) - P(T,t_m) + \sum_{i=1}^{m}(\alpha_i - 1)P\big(t,f_{i,s}\big) + \sum_{i=1}^{m}\left(\beta_i - \frac{\alpha_i - 1}{c_{i,f}}\right)c_{i,a}P(T,p_i)
\end{aligned} \tag{359}$$

where t_e and t_m are the effective and maturity date of the underlying swap. Using a cashflow representation for the underlying swap, we can denote the k-th cashflow by d_k and write the swaption price (358) in a more general form

$$V_{s,T}^{\mathrm{PS}} = P(s,T)\mathbb{E}_s^T\left[\left(\sum_k d_k P(T,T_k)\right)^+\right] \tag{360}$$

The sum of discounted swap cashflows comprises both floating and fixed leg. Under the constant multiplicative spread assumption, it has the form

$$\sum_k d_k P(T,T_k) = P(T,t_e) - P(T,t_m) + \sum_{i=1}^{m}(\eta_i - 1)P\big(T,f_{i,s}\big) - K\sum_{j=1}^{n} c_{j,a}P\big(T,p_j\big) \tag{361}$$

and under the constant additive spread assumption, it reads

$$\sum_k d_k P(T,T_k) = P(T,t_e) - P(T,t_m) + \sum_{i=1}^{m}\delta_i c_{i,a}P(T,p_i) - K\sum_{j=1}^{n} c_{j,a}P\big(T,p_j\big) \tag{362}$$

The forward bond dynamics under $\mathbb{Q}^T$ is a lognormal martingale, which is given in (351) and repeated here for convenience

$$P(T,T_k) = P(s,T,T_k)\exp\left(-\frac{1}{2}\xi_k^2 - \xi_k Z^T\right) \qquad \text{with} \qquad \xi_k^2 = B_{T,T_k}^2\varphi_{s,T,T,T} \tag{363}$$

Note that usually we have $\eta_i - 1 > 0$ (or $\delta_i > 0$), hence the argument (i.e., the d_k's are all positive (negative) up to a certain $k = p$ and then all negative (positive)) stated in section 0 may not hold. However, since $\eta_i - 1$ (or δ_i) is generally much smaller than notional 1, it is almost certain to have a unique solution z^*, such that

$$\sum_k d_k P(s,T_k)\exp\left(-\frac{1}{2}\xi_k^2 - \xi_k z^*\right) = 0 \tag{364}$$

The payer and receiver swaption can then be priced using formula (309) and (311) respectively (with z^* defined differently!), that is

$$V_{s,T}^{\mathrm{PS}} = \sum_k d_k P(s,T_k)\Phi(-z^* - \xi_k)\,, \qquad V_{s,T}^{\mathrm{RS}} = \sum_k d_k P(s,T_k)\Phi(z^* + \xi_k) \tag{365}$$

It should be noted that, in practice, underlying swap of a swaption with exercise tenor of T (e.g., mM period) is different from a swap forward starting in a period of T. Effective date of the former is computed as the swaption exercise date plus the swap spot lag, and the exercise date itself is computed as today plus the exercise tenor using the relevant calendar and the business day convention of the underlying swap (e.g., $t_e = t_0 \oplus T \oplus \Delta_s$). The later however has an effective date $t_e = t_0 \oplus \Delta_s \oplus T$. This may introduce a difference of a few days between the two swaps.

8.5.5. Finite Difference Method

Let us denote $U(t, x_t)$ the value of a derivative driven by the stochastic process x_t in (334). Under risk neutral measure, its evolution is governed by the drift-diffusion PDE (158) with drift $\mu_{t,x} = \varphi_{s,t,t,t} - \kappa x_t$, volatility $\sigma_{t,x} = \sigma_t$ and risk-free rate $r_{t,x} = f_{s,t} + x_t$, where the quantity $\varphi_{s,t,t,t}$ is given in (336). The PDE can be solved numerically using the finite difference method introduced in section 6.2.

Noting that the driving process x_t has a drift term due to continuous change of measure (as explained in Section 7.2). Such drift is generally not favored by finite difference method. Hence, we may re-express the short rate in terms of a risk neutral martingale process ω_t such that

$$\omega_t = x_t - \chi_{s,t,t,t} = \int_s^t E_{u,t}\sigma_u d\widetilde{W}_u\,, \qquad d\omega_t = -\kappa\omega_t dt + \sigma_t d\widetilde{W}_t$$
$$\omega_t = E_{v,t}\int_s^v E_{u,v}\sigma_u d\widetilde{W}_u + \int_v^t E_{u,t}\sigma_u d\widetilde{W}_u = E_{v,t}\omega_v + \int_v^t E_{u,t}\sigma_u d\widetilde{W}_u \tag{366}$$

with $\chi_{s,t,t,t}$ in (336). The ω_t is normally distributed with mean 0 and variance $\varphi_{s,t,t,t} = \int_s^t E_{u,t}^2\sigma_u^2 du$. With this definition, the zero coupon bond is given as

$$P_{t,T} = \frac{P_{s,T}}{P_{s,t}}\exp\left(-\frac{1}{2}B_{t,T}^2\varphi_{s,t,t,t} - B_{t,T}\chi_{s,t,t,t} - B_{t,T}\omega_t\right) \tag{367}$$

and the short rate (334) becomes

$$r_t = f_{s,t} + \chi_{s,t,t,t} + \omega_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \varphi_{s,t,t,t} - \kappa\chi_{s,t,t,t} - \kappa\omega_t\right)dt + \mathbb{1}'\sigma_t d\widetilde{W}_t \tag{368}$$

The derivative price $U(t, \omega_t)$ driven by the process ω_t is also governed by the drift-diffusion PDE (158) with drift $\mu_{t,x} = -\kappa x_t$, volatility $\sigma_{t,x} = \sigma_t$ and risk-free rate $r_{t,x} = f_{s,t} + \chi_{s,t,t,t} + \omega_t$,. Alternatively, the price $U(t, \omega_t^Z)$ can be evaluated under Z-forward measure associated with numeraire $P_{t,Z}$. The derivative payoff is contingent on the stochastic process ω_t^Z which has the form

$$\omega_t^Z = \int_s^t E_{u,t}\sigma_u dW_u^Z = E_{v,t}\omega_v^Z + \int_v^t E_{u,t}\sigma_u dW_u^Z = \omega_t + \chi_{s,t,t,Z}$$
$$d\omega_t^Z = -\kappa\omega_t^Z dt + \sigma_t dW_u^Z \tag{369}$$

through the change of measure (294), e.g., in 1D

$$dW_t^Z = d\widetilde{W}_t + B_{t,Z}\sigma_t dt \tag{370}$$

The dynamics of the numeraire can be derived from (292)

$$\frac{dP_{t,Z}}{P_{t,Z}} = r_t dt - B_{t,Z}\sigma_t d\widetilde{W}_t = \left(f_{s,t} - B_{t,Z}\varphi_{s,t,t,t} + \omega_t^Z + B_{t,Z}^2\sigma_t^2\right)dt - B_{t,Z}\sigma_t dW_t^Z \tag{371}$$

where the short rate (368) by (288) becomes

$$r_t = f_{s,t} + \chi_{s,t,t,t} + \omega_t = f_{s,t} - B_{t,Z}\varphi_{s,t,t,t} + \omega_t^Z, \qquad B_{t,Z}\varphi_{s,t,t,t} = \chi_{s,t,t,V} - \chi_{s,t,t,t} \tag{372}$$

Hence, the derivative price $U(t,\omega_t^Z)$ must follow the PDE in (157) with

$$a = -\frac{\sigma_t^2}{2}, \qquad b = -B_{t,Z}\sigma_t^2 + \kappa\omega_t^Z, \qquad c = f_{s,t} - B_{t,Z}\varphi_{s,t,t,t} + \omega_t^Z \tag{373}$$

The PDE can then be solved accordingly.

8.5.6. Monte Carlo Simulation

In simulation, the most essential ingredient is to simulate the state variable (can be multi-dimensional) that determines the state of the yield curves and the numeraire. American/Bermudan style options can be priced using least square Monte Carlo (LSMC) method proposed by Longstaff and Schwartz in 2001 [42].

Let $s = T_0 < \cdots < T_i < \cdots < T_n$ be an array of anchor dates (shown in Figure 8.1; e.g., exercise/cashflow/payoff payment dates). Firstly starting from $s = T_0$, the state variable $x_{i,j}$ is simulated in a forward manner for the i-th time step and j-th path based on the conditional distribution of x. The yield curves $\hat{P}_{i,j}(t,T)$ and $P_{i,j}(t,T)$ and the numeraire $N_{i,j}$ are then determined by the simulated $x_{i,j}$. Under different equivalent martingale measures, expressions for these quantities may differ. We will discuss these in detail for two measures that are commonly used in simulation: the risk neutral measure and the Z-forward measure.

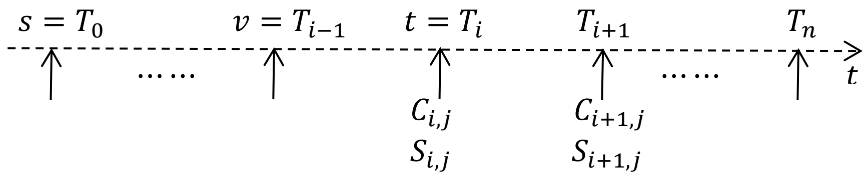


Figure 8.1 Two consecutive exercise dates of a Bermudan style option

Once the simulation paths are obtained, the LSMC method will be employed in a backward manner on the simulated paths. At exercise time T_i, there are two values for each path: 1) the immediate exercise payoff value $S_{i,j}$ estimated from $\hat{P}_{i,j}(t,T)$ and $P_{i,j}(t,T)$ and 2) the continuation value $C_{i,j}$ (i.e., the value of holding rather than exercising the option). The option holder compares the exercise payoff $S_{i,j}$ with the *conditional expectation* of the continuation value $\tilde{\mathbb{E}}_i[C_{i,j}]$ and exercise the option if the payoff value is higher. In the LSMC method, the conditional expectation, which is a function of state variable $x_{i,j}$, is approximated by a linear combination of basis functions

$$G(x_{i,j}) = \tilde{\mathbb{E}}_i[C_{i,j}] = \sum_k \hat{a}_k p_k(x_{i,j}) \tag{374}$$

where the basis functions $p_k(\cdot)$ can be a set of orthogonal polynomials (e.g., weighted Laguerre polynomials). In fact, as it has been pointed out, even simple polynomials $p_k(x) = x^k \ \forall\, k = 0,1,\cdots$ can serve the purpose very well. The factor loadings $\hat{a}_k$ are estimated by regressing $C_{i,j}$ on $p_k(x_{i,j})$ for all the in-the-money (i.e., exercising the option is economically advantageous) paths (denoted by I_i) using the linear model

$$C_{i,j} = \sum_k a_k p_k(x_{i,j}) + \epsilon_j \quad \forall\, j \in I_i \tag{375}$$

The out-of-the-money paths are excluded because they give the holder no choice but to keep holding it. This makes them less relevant to the estimation of the conditional expectation. The number of basis functions to be included in the regression (i.e., the upper bound for k) depends on the shape of the function

$G(x_{i,j})$. If the function is ill-shaped[1], higher degree of polynomials are desired to provide a better fit. Since the regression model is only used to make in-the-sample estimations, oscillation effect of higher degree of polynomials should not be an issue. In practice, the treatment of $S_{i,j}$ and $C_{i,j}$ differs from one product to another. Table 8.1 summarizes the formulas for two different products as examples, both have Bermudan style option embedded.

Table 8.1 Bermudan style interest rate products

Product	*Bermudan Swaption*	*Bermudan Cancellable Swap*[2]
Exercise Payoff Value[3]	$S_{i,j} = V_{i,n;j}$	$S_{i,j} = 0$
Continuation Value	$C_{i,j} = C_{i+1,j}\dfrac{N_{i,j}}{N_{i+1,j}}$	$C_{i,j} = V_{i,i+1;j} + C_{i+1,j}\dfrac{N_{i,j}}{N_{i+1,j}}$
In-the-money Paths	$I_i = \{j: V_{i,n;j} > 0\}$	$I_i = \{j: V_{i,i+1;j} < 0\}$

The backward evolution eventually leads to a present value of a product at time s for each simulation path. Average of the present values gives an estimate of the product price.

8.5.6.1. *Simulation under Risk Neutral Measure*

Under risk neutral measure, we know that the factor ω_t in (366) is normally distributed and the numeraire (i.e., money market account) M_t can be calculated as

$$M_t = \exp\left(\int_s^t r_u du\right) = \frac{1}{P_{s,t}} \exp\left(\int_s^t \chi_{s,v,v,v} dv + \int_s^t \omega_u du\right) \tag{376}$$

The integral $\int_s^t \chi_{s,v,v,v} dv$ in (376) is deterministic and can be calculated analytically as

$$\begin{aligned}\int_s^t \chi_{s,v,v,v} dv &= \int_s^t \int_s^v E_{u,v} B_{u,v} \sigma_u^2 du\, dv = \int_s^t \int_u^t E_{u,v} B_{u,v} dv\, \sigma_u^2 du = \frac{1}{2}\int_s^t B_{u,v}^2\Big|_{v=u}^t \sigma_u^2 du \\ &= \frac{1}{2}\int_s^t B_{u,t}^2 \sigma_u^2 du = \frac{1}{2}\psi_{s,t,t,t}\end{aligned} \tag{377}$$

[1] It is likely to have ill-shaped conditional expectation function if the option payoff is not simple. For example, the Bermudan cancellable range swaps may demand $k \geq 5$.
[2] Cancellable IRS can be regarded as a portfolio of a vanilla IRS plus a Bermudan swaption. For example, we may write: *Receiver IRS + Bermudan Payer Swaption = Receiver Cancellable Receiver IRS*
[3] The $V_{i,n;j}$ denotes the value of a swap starting at T_i and ending at T_n given state $x_{i,j}$.

with $\psi_{s,t,T,V}$ given in (336). Further, we define

$$\theta_t = \int_s^t \omega_h dh \tag{378}$$

Based on the conditional ω_t in (366), we can write the conditional θ_t given $\mathcal{F}_v$ for $s < v < t$ as

$$\begin{aligned}\theta_t &= \theta_v + \int_v^t \omega_h dh = \theta_v + \omega_v \int_v^t E_{v,h} dh + \int_v^t \int_v^h E_{u,h}\sigma_u d\widetilde{W}_u\, dh \\ &= \theta_v + B_{v,t}\omega_v + \int_v^t \int_u^t E_{u,h} dh\, \sigma_u d\widetilde{W}_u = \theta_v + B_{v,t}\omega_v + \int_v^t B_{u,t}\sigma_u d\widetilde{W}_u\end{aligned} \tag{379}$$

In summary, the x_t and y_t are jointly normally distributed with their conditional mean and variance as

$$\widetilde{\mathbb{E}}_v \begin{bmatrix}\omega_t \\ \theta_t\end{bmatrix} = \begin{bmatrix}E_{v,t} & 0 \\ B_{v,t} & 1\end{bmatrix}\begin{bmatrix}\omega_v \\ \theta_v\end{bmatrix}, \qquad \widetilde{\mathbb{V}}_v \begin{bmatrix}\omega_t \\ \theta_t\end{bmatrix} = \begin{bmatrix}\varphi_{v,t,t,t} & \chi_{v,t,t,t} \\ \chi_{v,t,t,t} & \psi_{v,t,t,t}\end{bmatrix} \tag{380}$$

while the conditional correlation between ω_t and θ_t is

$$\rho_{v,t} = \frac{\int_v^t E_{u,t} B_{u,t} \sigma_u^2 du}{\sqrt{\int_v^t E_{u,t}^2 \sigma_u^2 du \int_v^t B_{u,t}^2 \sigma_u^2 du}} = \frac{\chi_{v,t,t,t}}{\sqrt{\varphi_{v,t,t,t}\psi_{v,t,t,t}}} \tag{381}$$

In fact, the $\chi_{v,t,t,t}$ is also the covariance between the short rate r_t and the money market account M_t.

In simulation, the state variable (ω, θ) is actually in 2D. The variable ω drives the yield curves while the variable θ determines the numeraire. The simulation starts from $(\omega_s, \theta_s) = (0,0)$ and for each time step from $v = T_{i-1}$ to $t = T_i$, simulation generates (ω_t, θ_t) by (380)

$$\omega_t = E_{v,t}\omega_v + \sqrt{\varphi_{v,t,t,t}} B_{1,t}, \qquad \theta_t = \theta_v + B_{v,t}\omega_v + \sqrt{\psi_{v,t,t,t}}\left(\rho_{v,t} N_{1,t} + \sqrt{1 - \rho_{v,t}^2}\, B_{2,t}\right) \tag{382}$$

where $B_{1,t}$ and $B_{2,t}$ are two independent standard normal random samples and the correlation $\rho_{v,t}$ in (381). From the simulated (ω_t, θ_t), we can derive the yield curves by (346) (under constant multiplicative spread assumption) and the numeraire by (376), that is

$$\frac{\hat{P}_{t,T}}{\hat{P}_{s,t,T}} = \frac{P_{t,T}}{P_{s,t,T}} = \exp\left(-\frac{\xi_{s,t,t,T}^2}{2} - B_{t,T}\chi_{s,t,t,t} - B_{t,T}\omega_t\right), \qquad N_t = M_t = \frac{1}{P_{s,t}} \exp\left(\frac{\psi_{s,t,t,t}}{2} + \theta_t\right) \tag{383}$$

8.5.6.2. *Simulation under Z-forward measure*

Under Z-forward measure, the yield curves $\hat{P}_{t,T}$ and $P_{t,T}$ and the numeraire $P_{t,Z}$ are all driven by the common stochastic driver ω_t^Z defined in (369). The ω_t^Z has an initial value $\omega_s^Z = 0$ and is normally distributed with conditional mean and variance as

$$\mathbb{E}_v^Z[\omega_t^Z] = E_{v,t}\omega_v^Z, \qquad \mathbb{V}_v^Z[\omega_t^Z] = \varphi_{v,t,t,t} \tag{384}$$

Given the state of ω_t^Z, the yield curves and the numeraire can be derived from (346) and (369) as

$$\begin{aligned} \frac{\hat{P}_{t,T}}{\hat{P}_{s,t,T}} &= \frac{P_{t,T}}{P_{s,t,T}} = \exp\left(\left(B_{t,T}B_{t,Z} - \frac{B_{t,T}^2}{2}\right)\varphi_{s,t,t,t} - B_{t,T}\omega_t^Z\right) \\ N_t = P_{t,Z} &= \frac{P_{s,Z}}{P_{s,t}} \exp\left(\frac{B_{t,Z}^2\varphi_{s,t,t,t}}{2} - B_{t,Z}\omega_t^Z\right) = P_{s,t,Z} \exp\left(\frac{\xi_{s,t,t,Z}^2}{2} - B_{t,Z}\omega_t^Z\right) \end{aligned} \tag{385}$$

It should be emphasized that the volatility of the numeraire $P_{t,Z}$ increases as the maturity Z extends, hence in order to have faster convergence rate, it is better to have the Z no later than the trade maturity.

8.5.7. Range Accrual

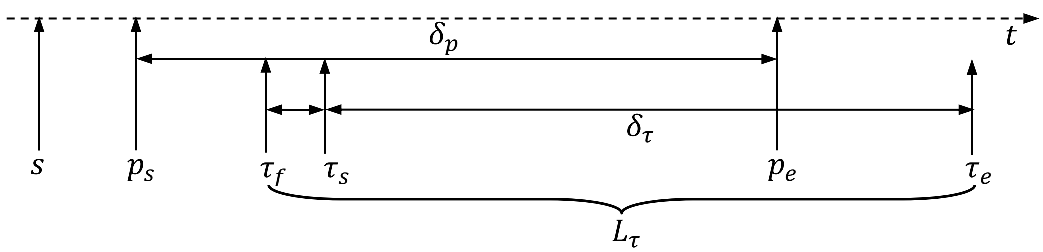


Figure 8.2 One coupon period of range accrual

This section is based on Hagan's work [43]. For a common range accrual, the fixed leg coupon payment depends on the number of days in the coupon period (e.g., between p_s and p_e as shown in Figure 8.2) having fixed Libor rates in a specific range, say $l \le L_\tau \le u \ \forall \tau \in [p_s, p_e]$. In other words, the period p coupon is determined by

$$C_p = \delta_p R \frac{\#\{\tau \in [p_s, p_e] :\ L_\tau \in [l, u]\}}{M_p}, \qquad M_p = \qquad \{\tau \in [p_s, p_e]\} \tag{386}$$

where δ_p is the coverage (year fraction) of the coupon period and R is the contractual fixed rate. The coupon payment can be valued by replicating each day's contribution in terms of vanilla caplets/floorlets and them summing over all days τ in the coupon period $[p_s, p_e]$. Suppose the day τ Libor rate L_τ is fixed at day τ_f for an effective (start) date τ_s and maturity (end) date τ_e with a coverage δ_τ. On the fixing date, the value of contribution from day τ is equal to the payoff

$$V_\tau(\tau_f, p_e) = P(\tau_f, p_e)\frac{\delta_p R}{M_p}\mathbb{1}\{L_\tau \in [l,u]\} \quad \text{where} \quad \mathbb{1}\{L_\tau \in [l,u]\} = \begin{cases} 1 & \text{if } L_\tau \in [l,u] \\ 0 & \text{otherwise} \end{cases} \tag{387}$$

Let us define the value at t of a digital floorlet L_τ with strike K as

$$\hat{F}_\tau(t,K) = P(t,\tau_e)\mathbb{E}_t^{\tau_e}[\mathbb{1}\{L_\tau \le K\}] = P(t,\tau_f)\mathbb{E}_t^{\tau_f}\left[P(\tau_f,\tau_e)\mathbb{1}\{L_\tau \le K\}\right] \tag{388}$$

If $L_\tau \le K$, the digital floorlet pays one unit of currency on the maturity of the Libor rate, otherwise pays nothing. So on the fixing date τ_f the payoff is known to be

$$\hat{F}_\tau(\tau_f, K) = P(\tau_f,\tau_e)\mathbb{1}\{L_\tau \le K\} \tag{389}$$

We can replicate the coupon payoff in (387) by going long and short digitals struck at l and u respectively, this yields

$$\frac{\delta_p R}{M_p}\left(\hat{F}_\tau(\tau_f,u) - \hat{F}_\tau(\tau_f,l)\right) = P(\tau_f,\tau_e)\frac{\delta_p R}{M_p}\mathbb{1}\{L_\tau \in [l,u]\} \tag{390}$$

This is the same payoff as in (387), except that the digitals pay off on τ_e instead of p_e.

Before fixing the date mismatch, we note that digitals are considered vanilla instruments because they can be replicated to arbitrary accuracy by a bullish spread of floorlets. Let us define the value at t of a standard floorlet on day τ Libor with strike K as

$$F_\tau(t,K) = P(t,\tau_e)\delta_\tau\mathbb{E}_t^{\tau_e}[(K-L_\tau)^+] = P(t,\tau_f)\delta_\tau\mathbb{E}_t^{\tau_f}\left[P(\tau_f,\tau_e)(K-L_\tau)^+\right] \tag{391}$$

So on the fixing date, the payoff is

$$F_\tau(\tau_f,K) = P(\tau_f,\tau_e)\delta_\tau(K-L_\tau)^+ \tag{392}$$

The bullish spread is constructed by going long $\frac{1}{2\varepsilon\delta_\tau}$ floorlets struck at $K^+ = K + \varepsilon$ and short the same number struck at $K^- = K - \varepsilon$. This yields the payoff

$$\frac{1}{2\varepsilon\delta_\tau}\Big(F_\tau\big(\tau_f,K^+\big)-F_\tau\big(\tau_f,K^-\big)\Big)=P\big(\tau_f,\tau_e\big)\cdot\begin{cases}1 & \text{if } L_k\le K^-\\ \dfrac{K+\varepsilon-L_k}{2\varepsilon} & \text{if } K^-<L_k\le K^+\\ 0 & \text{if } L_k>K^+\end{cases}\tag{393}$$

which goes to digital payoff as $\varepsilon\to 0$ (usually ε takes 5bps or 10bps). Hence, we have

$$\hat{F}_\tau\big(\tau_f,K\big)=\lim_{\varepsilon\to 0}\frac{F_\tau\big(\tau_f,K^+\big)-F_\tau\big(\tau_f,K^-\big)}{2\varepsilon\delta_\tau}\tag{394}$$

To handle the date mismatch, we rewrite the value of contribution in (387) as

$$V_\tau\big(\tau_f,p_e\big)=\frac{P\big(\tau_f,p_e\big)}{P\big(\tau_f,\tau_e\big)}P\big(\tau_f,\tau_e\big)\frac{\delta_p R}{M_p}\mathbb{1}\{L_\tau\in[l,u]\}\tag{395}$$

The ratio $P\big(\tau_f,p_e\big)/P\big(\tau_f,\tau_e\big)$ is the manifestation of the date mismatch. To handle the mismatch, we approximate the ratio by assuming the yield curve makes only parallel shifts over the relevant interval. Suppose we are at initial date $t=s$, then we assume that

$$\frac{\exp\big(-L_\tau(\tau_e-p_e)\big)}{P\big(\tau_f,p_e,\tau_e\big)}=\frac{\exp\Big(-L_{s,\tau}(\tau_e-p_e)\Big)}{P(s,p_e,\tau_e)}\Longrightarrow$$

$$\frac{P\big(\tau_f,p_e\big)}{P\big(\tau_f,\tau_e\big)}=\frac{P(s,p_e)}{P(s,\tau_e)}\frac{\exp\big(L_\tau(\tau_e-p_e)\big)}{\exp\Big(L_{s,\tau}(\tau_e-p_e)\Big)}\approx\frac{P(s,p_e)}{P(s,\tau_e)}\frac{1+L_\tau(\tau_e-p_e)}{1+L_{s,\tau}(\tau_e-p_e)}=\frac{P(s,p_e)}{P(s,\tau_e)}\frac{1+\eta\delta_\tau L_\tau}{1+\eta\delta_\tau L_{s,\tau}}\tag{396}$$

$$\text{where}\qquad \eta=\frac{\tau_e-p_e}{\tau_e-\tau_s}$$

This approximation accounts for the day count basis correctly (is exact when $p_e=\tau_s$) and is centered around the current forward value for the range coupon. With this approximation, the payoff from day τ becomes

$$\begin{aligned}&V_\tau\big(\tau_f,p_e\big)=P\big(\tau_f,p_e\big)\frac{\delta_p R}{M_p}\mathbb{1}\{L_\tau\in[l,u]\}=A_\tau P\big(\tau_f,\tau_e\big)(1+\eta\delta_\tau L_\tau)\mathbb{1}\{L_\tau\in[l,u]\}\\ &\text{where}\qquad A_\tau=\frac{P(s,p_e)}{P(s,\tau_e)}\frac{1}{1+\eta\delta_\tau L_{s,\tau}}\frac{\delta_p R}{M_p}\end{aligned}\tag{397}$$

The A_τ can be regarded as an effective notional fixed at date s. One can replicate the payoff in (397) by going long and short floorlet spreads centered around l and u.

To make it more general, let us consider

$$\begin{aligned} c_\tau(t,K,\varepsilon) &= \big(1+\eta\delta_\tau(K+\varepsilon)\big)C_\tau(t,K-\varepsilon) - \big(1+\eta\delta_\tau(K-\varepsilon)\big)C_\tau(t,K+\varepsilon) \\ f_\tau(t,K,\varepsilon) &= \big(1+\eta\delta_\tau(K-\varepsilon)\big)F_\tau(t,K+\varepsilon) - \big(1+\eta\delta_\tau(K+\varepsilon)\big)F_\tau(t,K-\varepsilon) \end{aligned} \tag{398}$$

where using the analogy of floorlet we define $C_\tau(t,K) = P(\tau_f,\tau_e)\delta_\tau(L_\tau-K)^+$ the value on date t of a standard caplet on day τ Libor rate with strike K, whose payoff is $C_\tau(\tau_f,K) = P(\tau_f,\tau_e)\delta_\tau(L_\tau-K)^+$.

Considering a floorlet spread

$$V_\tau^\varepsilon(t,p_e) = A_\tau \frac{f_\tau(t,K,\varepsilon)}{2\varepsilon\delta_\tau} \tag{399}$$

At time $t=\tau_f$, it has the payoff

$$V_\tau^\varepsilon(\tau_f,p_e) = A_\tau P(\tau_f,\tau_e)\frac{(1+\eta\delta_\tau K^-)(K^+-L_\tau)^+ - (1+\eta\delta_\tau K^+)(K^--L_\tau)^+}{2\varepsilon} \tag{400}$$

When $L_\tau > K^+$ the $V_\tau^\varepsilon(\tau_f,p_e)=0$ and when $L_\tau < K^-$ the payoff becomes

$$\begin{aligned} V_\tau^\varepsilon(\tau_f,p_e) &= A_\tau P(\tau_f,\tau_e)\frac{(1+\eta\delta_\tau K^-)(K^+-L_\tau) - (1+\eta\delta_\tau K^+)(K^--L_\tau)}{2\varepsilon} \\ &= A_\tau P(\tau_f,\tau_e)\frac{(K^+-K^-)+\eta\delta_\tau(K^+-K^-)L_\tau}{2\varepsilon} = A_\tau(\tau_f)P(\tau_f,\tau_e)(1+\eta\delta_\tau L_\tau) \end{aligned} \tag{401}$$

(The $V_\tau^\varepsilon(\tau_f,p_e)$ also has linear ramps for $L_\tau \in [K^-,K^+]$). Hence, in the limit $\varepsilon\to 0$, this is equivalent to

$$V_\tau^\varepsilon(\tau_f,p_e) = A_\tau P(\tau_f,\tau_e)(1+\eta\delta_\tau L_\tau)\mathbb{1}\{L_\tau < K\} \tag{402}$$

Similarly considering a caplet spread

$$V_\tau^\varepsilon(t,p_e) = A_\tau \frac{c_\tau(t,K,\varepsilon)}{2\varepsilon\delta_\tau} \tag{403}$$

At time $t=\tau_f$, it has the payoff

$$V_\tau^\varepsilon(\tau_f,p_e) = A_\tau P(\tau_f,\tau_e)\frac{(1+\eta\delta_\tau K^+)(L_\tau-K^-)^+ - (1+\eta\delta_\tau K^-)(L_\tau-K^+)^+}{2\varepsilon} \tag{404}$$

When $L_\tau < K^-$ the $V_\tau^\varepsilon(\tau_f,p_e)=0$ and when $L_\tau > K^+$ the payoff becomes

$$
\begin{aligned}
V_\tau^\varepsilon(\tau_f, p_e) &= A_\tau P(\tau_f, \tau_e) \frac{(1+\eta\delta_\tau K^+)(L_\tau - K^-) - (1+\eta\delta_\tau K^-)(L_\tau - K^+)}{2\varepsilon} \\
&= A_\tau P(\tau_f, \tau_e) \frac{(K^+ - K^-) + \eta\delta_\tau (K^+ - K^-) L_\tau}{2\varepsilon} = A_\tau P(\tau_f, \tau_e)(1+\eta\delta_\tau L_\tau)
\end{aligned}
\tag{405}
$$

Hence, in the limit $\varepsilon \to 0$, this is equivalent to

$$
V_\tau^\varepsilon(\tau_f, p_e) = A_\tau P(\tau_f, \tau_e)(1+\eta\delta_\tau L_\tau)\mathbb{1}\{L_\tau > K\}
\tag{406}
$$

Basically (399) and (403) are our building blocks. Using the caplet/floorlet spreads above, we are able to construct various rate ranges. For example, the floorlet spread portfolio

$$
V_\tau^\varepsilon(t, p_e) = A_\tau \frac{f_\tau(t,u,\varepsilon) - f_\tau(t,l,\varepsilon)}{2\varepsilon\delta_\tau} = A_\tau \frac{c_\tau(t,u,\varepsilon) - c_\tau(t,l,\varepsilon)}{2\varepsilon\delta_\tau}
\tag{407}
$$

has the payoff

$$
V_\tau^\varepsilon(\tau_f, p_e) = A_\tau P(\tau_f, \tau_e)(1+\eta\delta_\tau L_\tau)\mathbb{1}\{L_\tau \in [l,u]\}
\tag{408}
$$

which is the same as in (397).

The time t value of one period coupon payment of a range accrual is then given by summing the value contribution over all the days $\tau \in [p_s, p_e]$

$$
V_p(t) = \sum_\tau V_\tau^\varepsilon(t, p_e)
\tag{409}
$$

For vanilla range accrual swaps, the floorlets in (407) can be priced by Black model using implied caplet/floorlet volatility. Cancellable range accruals are usually valued in, for example, Hull-White model, where the floorlet price is given in (356), that is

$$
\begin{aligned}
&F_\tau(t,K) = V_{t,\tau}^{\mathrm{FLR}} = P(t,\tau_e)\left(-\frac{\hat{P}(t,\tau_s)}{\hat{P}(t,\tau_e)}\Phi(z^* + \xi_s) + (1+K\delta_\tau)\Phi(z^*+\xi_e)\right) \\
&z^* = \frac{\log\left((1+K\delta_\tau)\hat{P}(t,\tau_s,\tau_e)\right)}{\xi_e - \xi_s} - \frac{\xi_e - \xi_s}{2}, \qquad \xi_s = \xi(t,\tau_f,\tau_f,\tau_s), \qquad \xi_e = \xi(t,\tau_f,\tau_f,\tau_e)
\end{aligned}
\tag{410}
$$

We may write the present value of the coupon as

$$V_p(t) = P(t,p_e)\delta_p R\theta_p \qquad \text{where} \qquad \theta_p = \frac{1}{M_p}\sum_\tau \theta_\tau \tag{411}$$

The θ_p can be treated as an overall contribution coefficient, while the θ_τ comes from each day τ, which can be calculated as

$$\theta_\tau = \frac{V_\tau^\varepsilon(t,p_e)}{P(t,p_e)\dfrac{\delta_p R}{M_p}} = \frac{f_\tau(t,u,\varepsilon) - f_\tau(t,l,\varepsilon)}{2\varepsilon\delta_\tau P(t,\tau_e)\left(1+\eta\delta_\tau L_{t,\tau}\right)} \tag{412}$$

8.6. Historical Calibration of Hull-White Model via Kalman Filtering

The short rate model we want to calibrate has a general form derived from (295)

$$\begin{gathered} r_t = f_{s,t} + \mathbb{1}'\chi_{s,t,t,t}\mathbb{1} + \mathbb{1}'\omega_t, \quad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}'\left(\varphi_{s,t,t,t} - \kappa_t\chi_{s,t,t,t}\right)\mathbb{1} - \mathbb{1}'\kappa_t\omega_t\right)dt + \mathbb{1}'\sigma_t d\widetilde{W}_t \\ \omega_t = x_t - \chi_{s,t,t,t}\mathbb{1} = \int_s^t E_{u,t}\sigma_u d\widetilde{W}_u, \qquad d\omega_t = -\kappa_t\omega_t dt + \sigma_t d\widetilde{W}_t \end{gathered} \tag{413}$$

where the stochastic driver ω_t is a risk neutral martingale defined in the same manner as in (366), and the $\varphi_{s,t,T,V}$ and $\chi_{s,t,T,V}$ are given in (229).

8.6.1. <u>Market Price of Interest Rate Risk</u>

The data we use to calibrate our model are historical observations of yield curve. It is organized as a time series of zero rate term structure. The zero rate, defined as $Z_{t,\tau} = -\frac{1}{\tau}\log P_{t,t+\tau}$ for a maturity $\tau > 0$, can be expressed in Linear Gaussian model (i.e., the Hull-White model) by bond price (292)

$$\begin{aligned} P_{t,T} &= \frac{P_{s,T}}{P_{s,t}}\exp\left(-\frac{1}{2}\mathbb{1}'B_{t,T}\varphi_{s,t,t,t}B_{t,T}\mathbb{1} - \mathbb{1}'B_{t,T}\chi_{s,t,t,t}\mathbb{1} - \mathbb{1}'B_{t,T}\omega_t\right) \\ &= \frac{P_{s,T}}{P_{s,t}}\exp\left(-\mathbb{1}'\frac{\psi_{s,t,T,T} - \psi_{s,t,t,t}}{2}\mathbb{1} - \mathbb{1}'B_{t,T}\omega_t\right) \end{aligned} \tag{414}$$

with $\psi_{s,t,T,V}$ in (229). It should be noted that calibration to historical data differs from calibration to derivative prices. Derivatives must be priced under risk neutral measure to satisfy arbitrage-free condition, whereas historical data series are collected under physical measure. The change of measure from one to another can be done by introducing a market price of interest rate risk process λ_t, which is often assumed to be an affine function of ω_t, e.g., $\lambda_t = \eta_t + \delta_t \omega_t$ (this is equivalent to a popular assumption made by Vasicek [44] [45] [46] where the market price of risk is assumed to be an affine function of short rate r_t). Hence a Brownian motion $\widetilde{W}_t$ under risk neutral measure can be linked to a Brownian motion W_t under physical measure via

$$d\widetilde{W}_t = dW_t + \lambda_t dt = dW_t + (\eta_t + \delta_t \omega_t)dt \tag{415}$$

The ω_t in (413) under physical measure becomes

$$d\omega_t = \sigma_t \eta_t dt - (\kappa_t - \delta_t)\omega_t dt + \sigma_t dW_t, \qquad \omega_t = \int_s^t E_{u,t} G_{u,t} \sigma_u \eta_u du + \int_s^t E_{u,t} G_{u,t} \sigma_u dW_u \tag{416}$$

where $G_{t,T}$ is defined similarly as $E_{t,T}$

$$\underset{n\times n}{G_{t,T}} = \mathrm{Diag}\begin{bmatrix} \vdots \\ G_{i;t,T} \\ \vdots \end{bmatrix}, \qquad G_{i;t,T} \equiv \exp\left(\int_t^T \delta_{i;u} du\right) \tag{417}$$

Accordingly, the conditional formula for ω_t shows

$$\omega_t = E_{v,t} G_{v,t} \omega_v + \int_v^t E_{u,t} G_{u,t} \sigma_u \eta_u du + \int_v^t E_{u,t} G_{u,t} \sigma_u dW_u \tag{418}$$

In summary, the zero rate and the variable ω_t form a measurement and state transition system

$$\begin{aligned} &\text{Measurement:} \quad Z_{t,T} = -\frac{\log P_{t,T}}{T-t} = \frac{1}{T-t}\left(\log\frac{P_{s,t}}{P_{s,T}} + \mathbb{1}' \frac{\psi_{s,t,T,T} - \psi_{s,t,t,t}}{2} \mathbb{1} + \mathbb{1}' B_{t,T} \omega_t\right) \\ &\text{State transition:} \quad \omega_t = E_{v,t} G_{v,t} \omega_v + \int_v^t E_{u,t} G_{u,t} \sigma_u \eta_u du + \int_v^t E_{u,t} G_{u,t} \sigma_u dW_u \end{aligned} \tag{419}$$

which are well suited for Kalman filtering.

8.6.2. Kalman Filter

Kalman filter can be used to estimate model parameters in a state-space form

$$\begin{aligned} &\text{Measurement:} \quad \underset{n\times 1}{y_i} = \underset{n\times 1}{a_i} + \underset{n\times m}{H_i} \underset{m\times 1}{x_i} + \underset{n\times 1}{r_i}, \qquad r_i \sim \mathcal{N}\left(0, \underset{n\times n}{R_i}\right) \\ &\text{State transition:} \quad \underset{m\times 1}{x_i} = \underset{m\times 1}{c_i} + \underset{m\times m}{F_i} \underset{m\times 1}{x_{i-1}} + \underset{m\times 1}{q_i}, \qquad q_i \sim \mathcal{N}\left(0, \underset{m\times m}{Q_i}\right) \end{aligned} \tag{420}$$

where r_i and q_i are Gaussian white noise with covariance R_i and Q_i respectively. It is designed to filter out the desired true signal and the unobserved component from unwanted noises. The measurement system is observable. It describes the relationship between the observed variables y_i and the state variables x_i. The transition system is unobservable. It describes the dynamics of the state variables as formulated by vector c_i and matrix F_i. The vector r_i and q_i are innovations for measurement and transition system respectively. They are assumed to follow multivariate Gaussian distribution with zero mean and covariance matrix R_i and Q_i respectively.

The Kalman filter is a recursive estimator. This means that only the estimated state from the previous time step and the current measurement are needed to compute the estimate for the current state. Define the mean and variance of x_i conditioning on the observed measurements $y_0, y_1, \cdots, y_h$ for $h \le i$

$$
\begin{aligned}
E_{x,i|h} &= \mathbb{E}_h[x_i] = \mathbb{E}[x_i | y_0, y_1, \cdots, y_h] \\
V_{x,i|h} &= \mathbb{V}\left[x_i - E_{x,i|h}\right] = \mathbb{E}\left[\left(x_i - E_{x,i|h}\right)^2\right]
\end{aligned} \tag{421}
$$

The procedure generally consists of four steps:

1. Initialize the state vector:

   Since we do not know anything about $E_{x,0|0}$, we will make an assumption $x_0 \sim \mathcal{N}(\mu, \Sigma)$

$$
E_{x,0|0} = \mu, \qquad V_{x,0|0} = \Sigma \tag{422}
$$

2. Predict the *a priori* state vector for $h = i - 1$ and $i = 1,2,\cdots$:

$$
\begin{aligned}
E_{x,i|h} &= \mathbb{E}_h[x_i] = \mathbb{E}_h[c_i + F_i x_h + q_i] = c_i + F_i E_{x,h|h} \\
V_{x,i|h} &= \mathbb{V}\left[x_i - E_{x,i|h}\right] = \mathbb{V}\left[c_i + F_i x_h + q_i - c_i - F_i E_{x,h|h}\right] = F_i V_{x,h|h} F_i' + Q_i
\end{aligned} \tag{423}
$$

3. Forecast the measurement equation based on $E_{x,i|h}$ and $V_{x,i|h}$:

$$
\begin{aligned}
E_{y,i|h} &= \mathbb{E}_h[y_i] = \mathbb{E}_h[a_i + H_i x_i + r_i] = a_i + H_i E_{x,i|h} \\
V_{y,i|h} &= \mathbb{V}\left[y_i - E_{y,i|h}\right] = \mathbb{V}\left[a_i + H_i x_i + r_i - a_i - H_i E_{x,i|h}\right] = H_i V_{x,i|h} H_i' + R_i
\end{aligned} \tag{424}
$$

4. Update the inference to the state vector using measurement residual $z_i = y_i - E_{y,i|h}$ and Kalman gain $\underset{m\times n}{K_i}$:

$$E_{x,i|i} = E_{x,i|h} + K_i z_i \quad \text{and}$$

$$\begin{aligned} V_{x,i|i} &= \mathbb{V}\left[x_i - E_{x,i|i}\right] = \mathbb{V}\left[x_i - E_{x,i|h} - K_i\left(y_i - E_{y,i|h}\right)\right] = \mathbb{V}\left[(I - K_i H_i)\left(x_i - E_{x,i|h}\right) - K_i r_i\right] \\ &= (I - K_i H_i)\, V_{x,i|h}(I - K_i H_i)' + K_i R_i K_i' = V_{x,i|h} - 2K_i H_i V_{x,i|h} + K_i\left(H_i V_{x,i|h} H_i' + R_i\right)K_i' \\ &= (I - 2K_i H_i)V_{x,i|h} + K_i V_{y,i|h} K_i' \end{aligned} \tag{425}$$

The error in the *a posteriori* state estimation is $x_i - E_{x,i|i}$. We want to minimize the expected value of the square of the magnitude of this vector, i.e., $\mathbb{E}_i\left[\left\|x_i - E_{x,i|i}\right\|^2\right]$. This is equivalent to minimizing the trace of the *a posteriori* estimate covariance matrix $V_{x,i|i}$. By setting its first derivative to zero, we can derive the optimal Kalman gain $\widehat{K}_i$

$$\frac{\partial \mathrm{tr}\left(V_{x,i|i}\right)}{\partial K_i} = -2V_{x,i|h}H_i' + 2K_i V_{y,i|h} = 0 \quad \Longrightarrow \quad \widehat{K}_i = V_{x,i|h}H_i' V_{y,i|h}^{-1} \tag{426}$$

Calculation of $V_{y,i|h}^{-1}$ involves matrix inverse, however if the R_i^{-1} is available and the $V_{x,i|h}$ has a much smaller dimension than $V_{y,i|h}$, the $V_{y,i|h}^{-1}$ can be calculated in a more efficient way using *Sherman-Morrison-Woodbury* formula. Given the optimal $\widehat{K}_i$ in (426), the $V_{x,i|i}$ can be further simplified to

$$V_{x,i|i} = \left(I - 2\widehat{K}_i H_i\right)V_{x,i|h} + \widehat{K}_i V_{y,i|h}\widehat{K}_i' = \left(I - 2\widehat{K}_i H_i\right)V_{x,i|h} + V_{x,i|h}H_i'\widehat{K}_i' = \left(I - \widehat{K}_i H_i\right)V_{x,i|h} \tag{427}$$

We recursively generate the residual z_i and its covariance $V_{y,i|h}$ by stepping through the above procedure for $i = 1, \cdots, N$. The model parameters are then estimated through Maximum Likelihood Estimation (MLE) by maximizing the log likelihood function of the z_i time series:

$$\begin{aligned} l(\theta) &= \sum_{i=1}^{N} \log\left((2\pi)^{-\frac{n}{2}}\left|V_{y,i|h}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2} z_i' V_{y,i|h}^{-1} z_i\right)\right) \\ &= -\frac{nN \log(2\pi)}{2} + \frac{1}{2}\sum_{i=1}^{N}\left(-\log\left|V_{y,i|h}\right| - z_i' V_{y,i|h}^{-1} z_i\right) \end{aligned} \tag{428}$$

We can ignore the constant term and constant multiplier in front of the sum sign, hence maximizing the log likelihood function $l(\theta)$ is equivalent to maximizing the following sum:

$$\hat{l}(\theta) = \sum_{i=1}^{N}\left(-\log\left|V_{y,i|h}\right| - z_i' V_{y,i|h}^{-1} z_i\right) \tag{429}$$

8.6.3. Multi-Factor Hull-White Model

For effectiveness of calibration, the time-dependent coefficients κ_t, σ_t and the market price of risk λ_t are all assumed to be constant. The short rate in (413) becomes

$$\begin{aligned} & r_t = f_{s,t} + \mathbb{1}'\chi_{s,t,t,t}\mathbb{1} + \mathbb{1}'\omega_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \mathbb{1}'\left(\varphi_{s,t,t,t} - \kappa\chi_{s,t,t,t}\right)\mathbb{1} - \mathbb{1}'\kappa\omega_t\right)dt + \mathbb{1}'\sigma d\widetilde{W}_t \\ & \omega_t = \sigma\int_s^t E_{u,t} d\widetilde{W}_u, \qquad d\omega_t = -\kappa\omega_t dt + \sigma d\widetilde{W}_t \end{aligned} \tag{430}$$

Defining the maturity period $\tau = T - t$ and a timeline $s < \cdots < v < t < \cdots$, the measurement and state transition systems in (419) read

$$\begin{aligned} & Z_{t,\tau} = \frac{1}{\tau}\left(\log\frac{P_{s,t}}{P_{s,t+\tau}} + \mathbb{1}'\frac{\psi_{s,t,t+\tau,t+\tau} - \psi_{s,t,t,t}}{2}\mathbb{1}\right) + \mathbb{1}'\frac{B_{t,t+\tau}}{\tau}\omega_t + \epsilon_{t,\tau}, \qquad \epsilon_{t,\tau} \sim \mathcal{N}\left(0, \nu^2 d^{\frac{\tau}{\tau^*}}\right) \\ & \omega_t = B_{v,t}\sigma\lambda + E_{v,t}\omega_v + \varepsilon_{v,t}, \qquad \varepsilon_{v,t} \sim \mathcal{N}\left(0, \varphi_{v,t,t,t}\right) \end{aligned} \tag{431}$$

where we introduce a measurement innovation $\epsilon_{t,\tau}$ (e.g., the noises recorded in the zero rates). The $\epsilon_{t,\tau}$ is assumed to follow a normal distribution with variance $\nu^2 d^{\frac{\tau}{\tau^*}}$, where the d is a user-specified constant (a value between 0 and 1 emphasizing short maturities or a value greater than 1 emphasizing longer maturities). The τ^* can be thought of as a normalization factor (e.g 1.0 if τ is annualized) and the ν is a unit innovation volatility (a parameter to be estimated).

We want to calibrate the model parameters $(\rho, \kappa, \sigma, \lambda, \nu)$ to historical observations of yield curve term structure. Since both κ and σ are constant (diagonal) matrices, the variance/covariance terms defined in (229) further simplify to

$$\underset{n\times n}{E_{t,T}} = \mathrm{Diag}\begin{bmatrix}\vdots\\ E_{i;t,T}\\ \vdots\end{bmatrix}, \qquad E_{i;t,T} = e^{-\kappa_i(T-t)}, \qquad \underset{n\times n}{B_{t,T}} = \mathrm{Diag}\begin{bmatrix}\vdots\\ B_{i;t,T}\\ \vdots\end{bmatrix}, \qquad B_{i;t,T} = \frac{1-e^{-\kappa_i(T-t)}}{\kappa_i}$$

$$\begin{aligned}
\left[\varphi_{s,t,T,T}\right]_{i,j} &= \rho_{ij}\sigma_i\sigma_j\int_s^t E_{i;u,T}E_{j;u,T}du = \rho_{ij}\sigma_i\sigma_j\gamma\left(\kappa_i+\kappa_j\right)\\
\left[\chi_{s,t,T,T}\right]_{i,j} &= \rho_{ij}\sigma_i\sigma_j\int_s^t E_{i;u,T}B_{j;u,T}du = \rho_{ij}\sigma_i\sigma_j\frac{\gamma(\kappa_i)-\gamma\left(\kappa_i+\kappa_j\right)}{\kappa_j}\\
\left[\psi_{s,t,T,T}\right]_{i,j} &= \rho_{ij}\sigma_i\sigma_j\int_s^t B_{i;u,T}B_{j;u,T}du = \rho_{ij}\sigma_i\sigma_j\frac{t-s-\gamma(\kappa_i)-\gamma\left(\kappa_j\right)+\gamma\left(\kappa_i+\kappa_j\right)}{\kappa_i\kappa_j}\\
\text{where}\qquad \gamma(\kappa) &= \int_s^t e^{-\kappa(T-u)}du = \frac{e^{-\kappa(T-t)}-e^{-\kappa(T-s)}}{\kappa}
\end{aligned} \tag{432}$$

Suppose the time series of zero rate term structure are recorded at n tenors τ_k for $k = 1,2,\cdots,n$ and the factors are indexed by $p = 1,2,\cdots,m$, the (431) can be discretized by following the notation in (420), that is

$$\text{Measurement:}\quad \underset{n\times 1}{y_t} = \underset{n\times 1}{a_t} + \underset{n\times m}{H_t}\,\underset{m\times 1}{x_t} + \underset{n\times 1}{r_t}, \qquad r_t \sim \mathcal{N}\left(0,\ \underset{n\times n}{R_t}\right)$$

$$\text{State transition:}\quad \underset{m\times 1}{x_t} = \underset{m\times 1}{c_t} + \underset{m\times m}{F_t}\,\underset{m\times m}{x_v} + \underset{m\times m}{q_t}, \qquad q_t \sim \mathcal{N}\left(0,\ \underset{m\times m}{Q_t}\right)$$

$$\begin{aligned}
&[y_t]_k = Z_{t,\tau_k}, \qquad [a_t]_k = \frac{1}{\tau_k}\left(\log\frac{P_{s,t}}{P_{s,t+\tau_k}} + \frac{1}{2}\sum_{i,j}\left[\psi_{s,t,t+\tau_k,t+\tau_k}\right]_{i,j} - \frac{1}{2}\sum_{i,j}\left[\psi_{s,t,t,t}\right]_{i,j}\right)\\
&[H_t]_{k,p} = \frac{B_{p;t,t+\tau_k}}{\tau_k}, \qquad R_t = \mathrm{Diag}\begin{bmatrix}\vdots\\ d^{\frac{\tau_k}{\tau^*}}\nu^2\\ \vdots\end{bmatrix}\\
&x_t = \omega_t, \qquad c_t = B_{v,t}\sigma\lambda, \qquad F_t = E_{v,t}, \qquad Q_t = \varphi_{v,t,t,t}
\end{aligned} \tag{433}$$

Assuming the initial state $E_{x,s|s} = 0$ and $V_{x,s|s} = 0$, the parameters $(\rho,\kappa,\sigma,\lambda,\nu)$ can be estimated by Kalman filter introduced in section 8.6.2.

8.6.4. One-Factor Hull-White Model

Again we assume constant κ, σ and λ. In 1D, the short rate process in (430) simplifies to

$$
\begin{aligned}
&r_t = f_{s,t} + \chi_{s,t,t,t} + \omega_t, \qquad dr_t = \left(\frac{\partial f_{s,t}}{\partial t} + \varphi_{s,t,t,t} - \kappa\chi_{s,t,t,t} - \kappa\omega_t\right)dt + \sigma d\widetilde{W}_t \\
&\omega_t = \int_s^t E_{u,t}\sigma d\widetilde{W}_u, \qquad d\omega_t = -\kappa\omega_t dt + \sigma d\widetilde{W}_t
\end{aligned} \tag{434}
$$

Defining the maturity period $\tau = T - t$, the measurement and state transition system in (431) become

$$
\begin{aligned}
&Z_{t,\tau} = \frac{1}{\tau}\left(\log\frac{P_{s,t}}{P_{s,t+\tau}} + \frac{\psi_{s,t,t+\tau,t+\tau} - \psi_{s,t,t,t}}{2}\right) + \frac{B_{t,t+\tau}}{\tau}\omega_t + \epsilon_{t,\tau}, \qquad \epsilon_{t,\tau} \sim \mathcal{N}\left(0, \nu^2 d^{\frac{\tau}{\tau^*}}\right) \\
&\omega_t = B_{v,t}\sigma\lambda + E_{v,t}\omega_v + \varepsilon_{v,t}, \qquad \varepsilon_{v,t} \sim \mathcal{N}\left(0, \varphi_{v,t,t,t}\right)
\end{aligned} \tag{435}
$$

We want to calibrate the model parameters $(\kappa, \sigma, \lambda, \nu)$ to historical observations of yield curve term structure. Since both κ and σ are constant, we can simplify the variance/covariance in (432) to

$$
\begin{aligned}
&E_{t,T} = e^{-\kappa(T-t)}, \qquad B_{t,T} = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \\
&\varphi_{s,t,T,T} = \sigma^2\int_s^t E_{u,T}^2 du = \sigma^2\gamma(2\kappa) \\
&\chi_{s,t,T,T} = \sigma^2\int_s^t E_{u,T}B_{u,T} du = \sigma^2\frac{\gamma(\kappa) - \gamma(2\kappa)}{\kappa} \\
&\psi_{s,t,T,T} = \sigma^2\int_s^t B_{u,T}^2 du = \sigma^2\frac{t - s - 2\gamma(\kappa) + \gamma(2\kappa)}{\kappa^2} \\
&\text{where} \qquad \gamma(\kappa) = \int_s^t e^{-\kappa(T-u)} du = \frac{e^{-\kappa(T-t)} - e^{-\kappa(T-s)}}{\kappa}
\end{aligned} \tag{436}
$$

Suppose the time series of zero rate term structure are recorded at n tenors τ_k for $k = 1,2,\cdots,n$, following (433) we have

$$
\begin{aligned}
&\text{Measurement:} \quad \underset{n\times 1}{y_t} = \underset{n\times 1}{a_t} + \underset{n\times 1}{H_t}\,\underset{1\times 1}{x_t} + \underset{n\times 1}{r_t}, \qquad r_t \sim \mathcal{N}\left(0, \underset{n\times n}{R_t}\right) \\
&\text{State transition:} \quad x_t = c_t + F_t\, x_v + q_t, \qquad q_t \sim \mathcal{N}(0, Q_t) \\
&[y_t]_k = Z_{t,\tau_k}, \qquad [a_t]_k = \frac{1}{\tau_k}\left(\log\frac{P_{s,t}}{P_{s,t+\tau_k}} + \frac{\psi_{s,t,t+\tau_k,t+\tau_k} - \psi_{s,t,t,t}}{2}\right), \qquad [H_t]_k = \frac{B_{t,t+\tau_k}}{\tau_k}
\end{aligned} \tag{437}
$$

$$R_t = \text{Diag}\begin{bmatrix} \vdots \\ d^{\frac{\tau_k}{\tau^*}}\nu^2 \\ \vdots \end{bmatrix}, \qquad x_t = \omega_t, \qquad c_t = B_{v,t}\sigma\lambda, \qquad F_t = E_{v,t}, \qquad Q_t = \varphi_{v,t,t,t}$$

Assuming the initial state $E_{x,s|s} = 0$ and $V_{x,s|s} = 0$, the parameters $(\kappa, \sigma, \lambda, \nu)$ can be estimated by Kalman filter introduced in section 8.6.2.

8.7. Eurodollar Futures Rate Convexity Adjustment

8.7.1. General Formulas

We want to derive an analytical formula to estimate EDF Convexity adjustment in affine term structure models. For simplicity, let us temporarily omit the t variable in the subscripts and denote the bond price, for example, $P_{t,1}$ by P_1. The forward rate dynamics can then be derived by following (43)

$$\begin{aligned} df_{t,T,V} &= \frac{1}{\tau} d\frac{P_{t,T}}{P_{t,V}} = \frac{1}{\tau}\left(\frac{dP_{t,T}}{P_{t,V}} + P_{t,T} d\frac{1}{P_{t,V}} + dP_{t,T} d\frac{1}{P_{t,V}}\right) \\ &= \frac{P_{t,T}}{P_{t,V}\tau}\left(r_t dt - B_{t,T}\sigma_t d\widetilde{W}_t + B_{t,V}^2\sigma_t^2 dt - r_t dt + B_{t,V}\sigma_t d\widetilde{W}_t - B_{t,T}B_{t,V}\sigma_t^2 dt\right) \\ &= \frac{P_{t,T}\left(B_{t,V} - B_{t,T}\right)}{\tau P_{t,V}}\left(B_{t,V}^2\sigma_t^2 dt + \sigma_t d\widetilde{W}_t\right) \end{aligned} \tag{438}$$

Since

$$\frac{P_{t,T}}{\tau P_{t,V}} = \frac{1}{\tau} + f_{t,T,V} \tag{439}$$

we may also write

$$d\left(\frac{1}{\tau} + f_{t,T,V}\right) = \left(\frac{1}{\tau} + f_{t,T,V}\right)\left(B_{t,V} - B_{t,T}\right)\left(B_{t,V}^2\sigma_t^2 dt + \sigma_t d\widetilde{W}_t\right) \tag{440}$$

Because $f_{t,T,V}$ is given by a market tradable asset $(P_{t,T} - P_{t,V})$ denominated in a numeraire $P_{t,V}$ and then divided by a constant τ, according to (24) the $f_{t,T,V}$ is a martingale under V-forward measure $\mathbb{Q}^V$ associated with numeraire $P_{t,V}$. Since the volatility term remains the same after the change of numeraire, we can remove the drift term and write the forward rate dynamics under $\mathbb{Q}^V$ as

$$df_{t,T,V} = \frac{P_{t,T}\left(B_{t,V} - B_{t,T}\right)}{\tau P_{t,V}} \sigma_t dW_t^V \quad \text{or} \quad d\left(\frac{1}{\tau} + f_{t,T,V}\right) = \left(\frac{1}{\tau} + f_{t,T,V}\right)\left(B_{t,V} - B_{t,T}\right)\sigma_t dW_t^V \tag{441}$$

where $dW_t^V = d\widetilde{W}_t + B_{t,V}\sigma_t dt$ is a Brownian motion under $\mathbb{Q}^V$. This is consistent with the result implied from (23).

Let us denote the futures rate as $\not{f}_{t,T,V}$. The futures-forward spread (i.e., the convexity adjustment) $\Delta_s = \not{f}_{t,T,V} - f_{t,T,V}$ can be regarded as the accumulated difference in drift for the forward rate dynamics under two different probability measures, $\mathbb{Q}$ and $\mathbb{Q}^V$, respectively

$$\Delta_s = \widetilde{\mathbb{E}}_t\left[f_{T,T,V}\right] - \mathbb{E}_t^V\left[f_{T,T,V}\right] \tag{442}$$

Since $f_{t,T,V}$ is a martingale under $\mathbb{Q}^V$, the spread comes solely from the accumulated drift of $f_{t,T,V}$ under risk neutral measure $\mathbb{Q}$, that is

$$\Delta_s = \widetilde{\mathbb{E}}_t\left[f_{T,T,V}\right] - f_{t,T,V} \tag{443}$$

To calculate the quantity, we first integrate (440) from t to T

$$\begin{aligned} \frac{\frac{1}{\tau} + f_{T,T,V}}{\frac{1}{\tau} + f_{t,T,V}} = \exp\Bigg(&\int_t^T B_{u,V}\left(B_{u,V} - B_{u,T}\right)\sigma_u^2 du + \int_t^T \left(B_{u,V} - B_{u,T}\right)\sigma_u d\widetilde{W} \\ &- \frac{1}{2}\int_t^T \left(B_{u,V} - B_{u,T}\right)^2 \sigma_u^2 du\Bigg) \end{aligned} \tag{444}$$

then

$$\widetilde{\mathbb{E}}_t\left[f_{T,T,V}\right] = \left(\frac{1}{\tau} + f_{t,T,V}\right)\exp\left(\int_t^T B_{u,V}\left(B_{u,V} - B_{u,T}\right)\sigma_u^2 du\right) - \frac{1}{\tau} \tag{445}$$

Therefore

$$\Delta_s = \left(\frac{1}{\tau} + f_{t,T,V}\right)\left(\exp\left(\int_t^T B_{u,V}\left(B_{u,V} - B_{u,T}\right)\sigma_u^2 du\right) - 1\right) \tag{446}$$

This is formula for the convexity adjustment from EDF rate to FRA rate [47] [48] [49]. To simplify the above formula, we apply a few approximations. Firstly since τ is generally short and $f_{t,T,V}$ is much smaller than 1, we can assume $\left(1+\tau f_{t,T,V}\right)\approx 1$. Secondly if x is small, we may write $(e^x-1)\approx x$. Thus we have

$$\Delta_s \approx \frac{1}{\tau}\int_t^T B_{u,V}\left(B_{u,V}-B_{u,T}\right)\sigma_u^2 du \tag{447}$$

If we consider a continuously compounded forward rate $\zeta_{t,1,2}$ then

$$\zeta_{t,T,V} = \frac{\log P_{t,T} - \log P_{t,V}}{\tau} \tag{448}$$

and its dynamics can be derived as

$$\begin{aligned} d\zeta_{t,T,V} &= \frac{1}{\tau}\left(d\log P_{t,T} - d\log P_{t,V}\right) = \frac{1}{\tau}\left(\frac{dP_{t,T}}{P_{t,T}} - \frac{dP_{t,T}dP_{t,T}}{2P_{t,T}^2} - \frac{dP_{t,V}}{P_{t,V}} + \frac{dP_{t,V}dP_{t,V}}{2P_{t,V}^2}\right) \\ &= \frac{1}{\tau}\left(r_t dt - B_{t,T}\sigma_t d\widetilde{W}_t - \frac{1}{2}B_{t,T}^2\sigma_t^2 dt - r_t dt + B_{t,V}\sigma_t d\widetilde{W}_t + \frac{1}{2}B_{t,V}^2\sigma_t^2 dt\right) \\ &= \frac{B_{t,V}^2 - B_{t,T}^2}{2\tau}\sigma_t^2 dt + \frac{B_{t,V}-B_{t,T}}{\tau}\sigma_t d\widetilde{W}_t \end{aligned} \tag{449}$$

Since the $\zeta_{t,T,V}$ does not involve a zero coupon bond as numeraire, in theory it's not a martingale under V-forward measure. However, this property can still be reasonably assumed because firstly it is a quite accurate approximation and secondly the convexity is dominated by the drift of the forward rate under risk neutral measure. Therefore, we have the forward rate

$$\zeta_{t,T,V} \approx \mathbb{E}_t^V\left[\zeta_{T,T,V}\right] \tag{450}$$

Furthermore, we integrate (449) to have

$$\zeta_{T,T,V} = \zeta_{t,T,V} + \int_t^T \frac{B_{u,V}^2 - B_{u,T}^2}{2\tau}\sigma_u^2 du + \int_t^T \frac{B_{u,V}-B_{u,T}}{\tau}\sigma_u d\widetilde{W}_u \tag{451}$$

Then its expectation under risk neutral measure (i.e., the futures rate) becomes

$$\widetilde{\mathbb{E}}_t\left[\zeta_{T,T,V}\right] = \zeta_{t,T,V} + \int_t^T \frac{B_{u,V}^2 - B_{u,T}^2}{2\tau}\sigma_u^2 du \tag{452}$$

Eventually we reach the convexity adjustment formula for a continuously compounded forward rate

$$\Delta_c = \widetilde{\mathbb{E}}_t\left[\zeta_{T,T,V}\right] - \mathbb{E}_t^V\left[\zeta_{T,T,V}\right] \approx \frac{1}{2\tau}\int_t^T \left(B_{u,V}^2 - B_{u,T}^2\right)\sigma_u^2 du \tag{453}$$

Knowing the functional form of $B_{t,T}$ and σ_t in a specific model, the convexity adjustment for EDF can be calculated by (447) or (453) accordingly.

8.7.2. Convexity Adjustment in the Hull-White Model

With constant mean reversion rate κ, we have $B_{t,T} = \frac{1-e^{-\kappa(T-t)}}{\kappa}$. The convexity adjustment of EDF in Hull-White model can then be estimated by (447) if the forward rate is simply compounded

$$\begin{aligned}
\Delta_s &\approx \frac{1}{\tau}\int_t^T B_{u,V}\left(B_{u,V} - B_{u,T}\right)\sigma^2 du = \frac{\sigma^2}{\tau} B_{T,V}\int_t^T B_{u,V}E_{u,T} du \\
&= \frac{\sigma^2}{\tau\kappa} B_{T,V}\left(\int_t^T E_{u,T} du - \int_t^T E_{u,V}E_{u,T} du\right) = \frac{\sigma^2}{\tau\kappa} B_{T,V}\left(B_{t,T} - E_{T,V}\int_t^T E_{u,T}^2 du\right) \\
&= \frac{\sigma^2}{\tau\kappa} B_{T,V}\left(B_{t,T} - E_{T,V}\frac{1-E_{t,T}^2}{2\kappa}\right) = \frac{\sigma^2}{2\kappa\tau} B_{T,V}B_{t,T}\left(2 - E_{T,V} - E_{t,V}\right) \\
&= \frac{\sigma^2}{2\tau} B_{T,V}B_{t,T}\left(B_{T,V} + B_{t,V}\right)
\end{aligned} \tag{454}$$

or by (453) if the rate is continuously compounded [50] [51]

$$\begin{aligned}
\Delta_c &\approx \frac{1}{2\tau}\int_t^T \left(B_{u,V}^2 - B_{u,T}^2\right)\sigma^2 du = \frac{\sigma^2}{2\tau} B_{T,V}\int_t^T \left(B_{u,V} + B_{u,T}\right)E_{u,T} du \\
&= \frac{\sigma^2}{2\tau} B_{T,V}\left(\int_t^T B_{u,V}E_{u,T} du + \int_t^T B_{u,T}E_{u,T} du\right) \\
&= \frac{\sigma^2}{4\tau} B_{T,V}B_{t,T}\left(B_{T,V} + B_{t,V}\right) + \frac{\sigma^2}{2\kappa\tau} B_{T,V}\left(\int_t^T E_{u,T} du - \int_t^T E_{u,T}^2 du\right) \\
&= \frac{\sigma^2}{4\tau} B_{T,V}B_{t,T}\left(B_{T,V} + B_{t,V}\right) + \frac{\sigma^2}{2\kappa\tau} B_{T,V}\left(B_{t,T} - \frac{1-E_{t,T}^2}{2\kappa}\right) \\
&= \frac{\sigma^2}{4\tau} B_{T,V}B_{t,T}\left(B_{T,V} + B_{t,V}\right) + \frac{\sigma^2}{2\kappa\tau} B_{T,V}B_{t,T}\left(\frac{1-E_{t,T}}{2}\right) \\
&= \frac{\sigma^2}{4\tau} B_{T,V}B_{t,T}\left(B_{T,V} + B_{t,V} + B_{t,T}\right) = \frac{B_{T,V}\sigma^2}{4\tau\kappa}\left[B_{T,V}\left(1 - e^{-2\kappa(T-t)}\right) + 2\kappa B_{t,T}^2\right]
\end{aligned} \tag{455}$$

In a special case of $\kappa = 0$, the Hull-White model reduces to Ho-Lee Model, which has the $\mathbb{Q}$ dynamics of the spot rate

$$dr_t = \theta_t dt + \sigma d\widetilde{W}_t \tag{456}$$

Ho-Lee model is also an affine term structure model with

$$B_{t,T} = T - t \qquad \text{and} \qquad A_{t,T} = -\frac{\sigma^2 (T-t)^3}{6} + \int_t^T (T-u)\theta_u du \tag{457}$$

The convexity adjustment in Ho-Lee model can then be estimated as a special case of (454) if the forward rate is simply compounded

$$\Delta_s \approx \frac{1}{2}\sigma^2 (T_1 - t)(2T_2 - T_1 - t) \tag{458}$$

or by (455) if the rate is continuously compounded [49] [51]

$$\Delta_c \approx \frac{1}{2}\sigma^2 (T_1 - t)(T_2 - t) \tag{459}$$

9. THREE-FACTOR MODELS FOR FX AND INFLATION

In this chapter, we are going to discuss a 3-factor model, which has been widely used in both FX [52] and inflation market due to its simplicity and analytical tractability. In inflation market, this model is also called Jarrow-Yildirim model [53].

9.1. FX and Inflation Analogy

Table 9.1 Inflation and FX Analogy

Inflation Markets	*FX Markets*	*Notation/Definition*
nominal short rate	domestic short rate	r_t
nominal forward rate	domestic forward rate	$f_{t,T}$
nominal bond value in currency units	domestic bond value in domestic currency	$P_{t,T}$
real short rate	foreign short rate[1]	$\hat{r}_t$
real instantaneous forward rate	foreign instantaneous forward rate	$\hat{f}_{t,T}$
real bond value in inflation index units	foreign bond value in foreign currency	$\hat{P}_{t,T}$
inflation instantaneous forward rate	domestic minus foreign instantaneous forward rate[2]	$\bar{f}_{t,T} = f_{t,T} - \hat{f}_{t,T}$
inflation short rate	domestic minus foreign short rate (rate spread)	$\bar{r}_t = r_t - \hat{r}_t$
Inflation index spot level	spot FX rate (domestic ccy. per unit of foreign ccy.)[3]	x_t
TIPS Price: nominal value of a real bond	foreign bond value in domestic currency	$x_t\hat{P}_{t,T}$
forward index level	forward FX rate	$y_{t,T} = x_t \dfrac{\hat{P}_{t,T}}{P_{t,T}}$

The inflation market and FX market share a great similarity. For example, the domestic/foreign economy and the exchange rate in FX world are in analogy to the nominal/real economy and the inflation index rate, respectively, in inflation world. Table 9.1 shows the comparison between inflation and FX and their counterparts. In the following of this chapter, we will develop the model primarily in FX world

[1] A "hat" is used to denote a quantity in foreign currency/real economy.
[2] A "bar" is used to denote a quantity in relation to rate spread (discussed later).
[3] The exchange rate is expressed in a direct quotation format, e.g. a quote in fixed units of foreign currency against variable amounts of the domestic currency.

because it is less abstract and more straightforward to understand. However, the framework we build for FX can be seamlessly migrated and adapted for the inflation markets.

9.2. Three-factor Model: Modeling Short Rates

For short-dated FX option, we often assume deterministic domestic and foreign interest rates. The importance of interest rate risk grows as the FX option maturities increase. For pricing long-dated FX options, however this assumption is inadequate. We must develop a model that can sufficiently describe the interest rates dynamics together with the FX rate.

9.2.1. Model Definition

Suppose in our 3-factor model, the domestic forward rate $f_{t,T}$, the foreign forward rate $\hat{f}_{t,T}$ and the foreign exchange (FX) spot rate x_t (i.e., x_t units of domestic currency per one unit of foreign currency) are modeled by the following 3-factor SDE (the accent "hat" here denotes a quantity related to foreign economy)

$$df_{t,T} = \alpha_{t,T}dt + \beta'_{t,T}dW_t, \qquad d\hat{f}_{t,T} = \hat{\alpha}_{t,T}dt + \hat{\beta}'_{t,T}dW_t, \qquad \frac{dx_t}{x_t} = \mu_t dt + \delta'_t dW_t \tag{460}$$

where dW_t is a 3D standard Brownian motion (with independent components). The use of a multi-dimensional and independent stochastic driver here simplifies the handling of correlation structure, which is implied in the covariances (i.e., the dot product of two volatility vectors). For instance, the instantaneous covariance between the domestic and foreign forward rate can be written as $\beta'_{t,T}\hat{\beta}_{t,T}$, a dot product between two volatility vectors. This treatment avoids a whole set of explicit correlation parameters and makes the formulas and equations concise. Occasionally with abuse of notation, we may write, for example, $\beta^2_{t,T} \equiv \beta'_{t,T}\beta_{t,T}$ to denote a variance term.

9.2.2. Domestic Risk Neutral Measure

The first step is to rewrite the model under domestic risk neutral measure. We define

$$M_t = \exp\left(\int_s^t r_u du\right), \qquad \widehat{M}_t = \exp\left(\int_s^t \hat{r}_u du\right) \tag{461}$$

to be the domestic and foreign money market account in their own currency. The change from the physical measure $\mathbb{P}$ to the domestic risk neutral measure $\mathbb{Q}$ (associated with the numeraire M_t) is achieved as in (23) by a 3D vector of market price of risk λ_t such that

$$d\widetilde{W}_t = dW_t + \lambda_t dt \tag{462}$$

is a 3D Brownian motion under $\mathbb{Q}$. The λ_t can be uniquely determined (see below) by considering three market tradable assets: 1) the domestic bond $P_{t,T}$, 2) the foreign money market account $x_t\widehat{M}_t$ and 3) the foreign bond $x_t\widehat{P}_{t,T}$. These three assets when denominated in M_t are $\mathbb{Q}$-martingales.

We begin with domestic and foreign bond dynamics under physical measure given in (216)

$$\begin{aligned}\frac{dP_{t,T}}{P_{t,T}} &= \left(r_t - a_{t,T} + \frac{1}{2}b_{t,T}^2\right)dt - b'_{t,T}dW_t, \qquad a_{t,T} = \int_t^T \alpha_{t,u}du\,, \qquad b_{t,T} = \int_t^T \beta_{t,u}du \\ \frac{d\widehat{P}_{t,T}}{\widehat{P}_{t,T}} &= \left(\hat{r}_t - \hat{a}_{t,T} + \frac{1}{2}\hat{b}_{t,T}^2\right)dt - \hat{b}'_{t,T}dW_t, \qquad \hat{a}_{t,T} = \int_t^T \hat{\alpha}_{t,u}du\,, \qquad \hat{b}_{t,T} = \int_t^T \hat{\beta}_{t,u}du\end{aligned} \tag{463}$$

(Note that the $\mathbb{1}$ and ρ are omitted because the components of dW_t are independent and the covariance can be denoted by a dot product.) To determine the λ_t, we firstly consider the $\mathbb{Q}$-martingale $P_{t,T}M_t^{-1}$, whose dynamics is

$$\begin{aligned}\frac{d\left(P_{t,T}M_t^{-1}\right)}{P_{t,T}M_t^{-1}} &= \frac{dP_{t,T}}{P_{t,T}} + \frac{dM_t^{-1}}{M_t^{-1}} + \frac{dP_{t,T}}{P_{t,T}}\frac{dM_t^{-1}}{M_t^{-1}} = \left(-a_{t,T} + \frac{1}{2}b_{t,T}^2\right)dt - b'_{t,T}dW_t \\ &= \left(b'_{t,T}\lambda_t - a_{t,T} + \frac{1}{2}b_{t,T}^2\right)dt - b'_{t,T}d\widetilde{W}_t\end{aligned} \tag{464}$$

Following our previous derivation in (219), the drift term must vanish, we see that the first equation that λ_t must satisfy is

$$b'_{t,T}\lambda_t = a_{t,T} - \frac{1}{2}b_{t,T}^2 \qquad \text{or} \qquad \beta'_{t,T}\lambda_t = \alpha_{t,T} - \beta'_{t,T}b_{t,T} \tag{465}$$

Secondly, we consider the dynamics of the $\mathbb{Q}$-martingale $x_t\widehat{M}_tM_t^{-1}$

$$\frac{d\left(x_t\widehat{M}_tM_t^{-1}\right)}{x_t\widehat{M}_tM_t^{-1}} = \frac{dx_t}{x_t} + \frac{d\widehat{M}_t}{\widehat{M}_t} + \frac{dM_t^{-1}}{M_t^{-1}} = (\mu_t + \hat{r}_t - r_t)dt + \delta'_t dW_t \tag{466}$$

$$= (\mu_t + \hat{r}_t - r_t - \delta_t'\lambda_t)dt + \delta_t' d\widetilde{W}_t$$

Since the dirft term must vanish under $\mathbb{Q}$, we have the second equation that λ_t must hold for

$$\delta_t'\lambda_t = \mu_t + \hat{r}_t - r_t \tag{467}$$

Lastly, we consider the dynamics of the $\mathbb{Q}$-martingale $x_t\hat{P}_{t,T}M_t^{-1}$, which can be written as

$$\begin{aligned} \frac{d\left(x_t\hat{P}_{t,T}M_t^{-1}\right)}{x_t\hat{P}_{t,T}M_t^{-1}} &= \frac{dx_t}{x_t} + \frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} + \frac{dM_t^{-1}}{M_t^{-1}} + \frac{dx_t}{x_t}\frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} \\ &= \left(\mu_t + \hat{r}_t - \hat{a}_{t,T} + \frac{1}{2}\hat{b}_{t,T}^2 - r_t - \delta_t'\hat{b}_{t,T}\right)dt + \left(\delta_t - \hat{b}_{t,T}\right)' dW_t \\ &= \left(\delta_t'\lambda_t - \hat{a}_{t,T} + \frac{1}{2}\hat{b}_{t,T}^2 - \delta_t'\hat{b}_{t,T} - \left(\delta_t - \hat{b}_{t,T}\right)'\lambda_t\right)dt + \left(\delta_t - \hat{b}_{t,T}\right)' d\widetilde{W}_t \\ &= \left(-\hat{a}_{t,T} + \frac{1}{2}\hat{b}_{t,T}^2 - \delta_t'\hat{b}_{t,T} + \hat{b}_{t,T}'\lambda_t\right)dt + \left(\delta_t - \hat{b}_{t,T}\right)' d\widetilde{W}_t \end{aligned} \tag{468}$$

Since the drift term must be zero under $\mathbb{Q}$, we have the third equation that must hold for λ_t

$$\hat{b}_{t,T}'\lambda_t = \hat{a}_{t,T} - \frac{1}{2}\hat{b}_{t,T}^2 + \delta_t'\hat{b}_{t,T} \qquad \text{or} \qquad \hat{\beta}_{t,T}'\lambda_t = \hat{\alpha}_{t,T} - \hat{\beta}_{t,T}'\left(\hat{b}_{t,T} - \delta_t\right) \tag{469}$$

Given all the three equations we have derived, the λ_t can be uniquely determined, therefore the $\mathbb{Q}$ is unique and the market is complete. We summarize our results as follows

$$\begin{aligned} & d\widetilde{W}_t = dW_t + \lambda_t dt \qquad \text{and} \\ & \beta_{t,T}'\lambda_t = \alpha_{t,T} - \beta_{t,T}'b_{t,T}, \qquad \delta_t'\lambda_t = \hat{r}_t - r_t + \mu_t, \qquad \hat{\beta}_{t,T}'\lambda_t = \hat{\alpha}_{t,T} - \hat{\beta}_{t,T}'\left(\hat{b}_{t,T} - \delta_t\right) \end{aligned} \tag{470}$$

Hence, we can write the model in (460) under $\mathbb{Q}$ as

$$\begin{aligned} & df_{t,T} = \beta_{t,T}'b_{t,T}dt + \beta_{t,T}'d\widetilde{W}_t, \qquad d\hat{f}_{t,T} = \hat{\beta}_{t,T}'\left(\hat{b}_{t,T} - \delta_t\right)dt + \hat{\beta}_{t,T}'d\widetilde{W}_t \\ & \frac{dx_t}{x_t} = (r_t - \hat{r}_t)dt + \delta_t' d\widetilde{W}_t \end{aligned} \tag{471}$$

9.2.3. Change of Measure: from Foreign to Domestic

In analogy to domestic forward rate in (471), we can also write the foreign forward rate under foreign risk neutral measure $\hat{\mathbb{Q}}$ as

$$d\hat{f}_{t,T} = \hat{\beta}'_{t,T}\hat{b}_{t,T}dt + \hat{\beta}'_{t,T}d\hat{\widetilde{W}}_t \tag{472}$$

where $d\hat{\widetilde{W}}_t$ is a 3D standard Brownian motion under $\hat{\widetilde{\mathbb{Q}}}$. Comparing this equation with the foreign forward rate equation in (471), we can easily conclude that the change of measure from $\hat{\widetilde{\mathbb{Q}}}$ to $\widetilde{\mathbb{Q}}$ can be done by

$$d\hat{\widetilde{W}}_t = d\widetilde{W}_t - \delta_t dt \tag{473}$$

To make it more explanatory, we provide another way to view the change of measure. Given the two money market accounts M_t and $\widehat{M}_t$, if $\hat{C}_t$ is the value of a financial product in foreign currency, the no-arbitrage formula (24) tells

$$\begin{aligned} & x_t\widehat{M}_t\hat{\widetilde{\mathbb{E}}}_t\left[\frac{\hat{C}_T}{\widehat{M}_T}\right] = x_t\hat{C}_t = C_t = M_t\widetilde{\mathbb{E}}_t\left[\frac{x_T\hat{C}_T}{M_T}\right] \\ \Rightarrow\ & \hat{\widetilde{\mathbb{E}}}_t\left[\frac{x_t\hat{C}_T}{\widehat{M}_T}\right] = \frac{M_t}{\widehat{M}_t}\widetilde{\mathbb{E}}_t\left[\frac{x_T\hat{C}_T}{M_T}\right] = \widetilde{\mathbb{E}}_t\left[\frac{x_t\hat{C}_T}{\widehat{M}_T}\frac{d\hat{\widetilde{\mathbb{Q}}}}{d\widetilde{\mathbb{Q}}}\right] \Rightarrow \frac{d\hat{\widetilde{\mathbb{Q}}}}{d\widetilde{\mathbb{Q}}} = \frac{\widehat{M}_T}{\widehat{M}_t}\left(\frac{\frac{M_T}{x_T}}{\frac{M_t}{x_t}}\right)^{-1} \end{aligned} \tag{474}$$

In analogy to (25), the (474) shows that the change of measure from $\hat{\widetilde{\mathbb{Q}}}$ to $\widetilde{\mathbb{Q}}$ corresponds to the change of numeraire from $\widehat{M}_t$ to M_t/x_t, while reverting from $\widetilde{\mathbb{Q}}$ to $\hat{\widetilde{\mathbb{Q}}}$ corresponds to the change of numeraire from M_t to $x_t\widehat{M}_t$. Since $d\widehat{M}_t$ has nil volatility and the volatility of $d(M_t/x_t)$ is $-\delta_t$, based on (30) we reach the same result as in (473).

More explicitly, given that $x_t\widehat{M}_t$ is a tradable asset denominated in domestic currency, we can see that $x_t\widehat{M}_t/M_t$ must be a martingale under domestic risk neutral measure. Since the dx_t has a volatility of δ_t and the dM_t and $d\widehat{M}_t$ have nil volatilities, we can derive

$$\begin{aligned} & \frac{d\left(x_t\widehat{M}_tM_t^{-1}\right)}{x_t\widehat{M}_tM_t^{-1}} = \frac{dx_t}{x_t} + \frac{d\widehat{M}_t}{\widehat{M}_t} + \frac{dM_t^{-1}}{M_t^{-1}} = \frac{dx_t}{x_t} + \hat{r}_tdt - r_tdt = \delta'_td\widetilde{W}_t \\ \Rightarrow\ & \frac{dx_t}{x_t} = (r_t - \hat{r}_t)dt + \delta'_td\widetilde{W}_t \end{aligned} \tag{475}$$

Similarly, providing that M_T/x_t is a tradable asset denominated in foreign currency, the quantity $M_Tx_t^{-1}/\widehat{M}_t$ must be a martingale under foreign risk neutral measure, that is

$$
\begin{aligned}
&\frac{dM_T x_t^{-1}\widehat{M}_t^{-1}}{M_T x_t^{-1}\widehat{M}_t^{-1}} = \frac{dM_T}{M_T} + \frac{dx_t^{-1}}{x_t^{-1}} + \frac{d\widehat{M}_t^{-1}}{\widehat{M}_t^{-1}} \\
&= r_t dt - (r_t - \hat{r}_t)dt - \delta_t' d\widetilde{W}_t + \delta_t'\delta_t dt - \hat{r}_t dt \\
&= -\delta_t'\left(d\widetilde{W}_t - \delta_t dt\right) = -\delta_t' d\widehat{\widetilde{W}}_t \\
&\Longrightarrow \quad d\widehat{\widetilde{W}}_t = d\widetilde{W}_t - \delta_t dt
\end{aligned} \tag{476}
$$

where the dynamics of the exchange rate inverse is

$$
\frac{dx_t^{-1}}{x_t^{-1}} = -\frac{dx_t}{x_t} + \frac{dx_t}{x_t}\cdot\frac{dx_t}{x_t} = \frac{1}{x_t}\left(-(r_t - \hat{r}_t)dt - \delta_t' d\widetilde{W}_t + \delta_t'\delta_t dt\right) \tag{477}
$$

9.2.4. The Hull-White Model

If we assume the forward rate volatilities are in the form similar to (285)

$$
\begin{aligned}
&\beta_{t,T} = E_{t,T}\sigma_t, \qquad b_{t,T} = B_{t,T}\sigma_t, \qquad E_{t,T} = \exp\left(-\int_t^T \kappa_u du\right), \qquad B_{t,T} = \int_t^T E_{t,u} du \\
&\hat{\beta}_{t,T} = \widehat{E}_{t,T}\hat{\sigma}_t, \qquad \hat{b}_{t,T} = \widehat{B}_{t,T}\hat{\sigma}_t, \qquad \widehat{E}_{t,T} = \exp\left(-\int_t^T \hat{\kappa}_u du\right), \qquad \widehat{B}_{t,T} = \int_t^T \widehat{E}_{t,u} du
\end{aligned} \tag{478}
$$

where σ_t and $\hat{\sigma}_t$ are the short rate volatilities (in 3D), the domestic and foreign short rate r_t and $\hat{r}_t$ are then described by the Hull-White model. Based on (291) and the change of measure (473), the 3-factor model follows the dynamics below under the domestic risk neutral measure $\mathbb{Q}$

$$
\begin{aligned}
&r_t = f_{s,t} + \int_s^t \beta_{u,t}' b_{u,t} du + z_t, \qquad z_t = \int_s^t \beta_{u,t}' d\widetilde{W}_u, \qquad dz_t = -\kappa_t z_t dt + \sigma_t' d\widetilde{W}_t \\
&\hat{r}_t = \hat{f}_{s,t} + \int_s^t \hat{\beta}_{u,t}'\left(\hat{b}_{u,t} - \delta_u\right)du + \hat{z}_t, \qquad \hat{z}_t = \int_s^t \hat{\beta}_{u,t}' d\widetilde{W}_u, \qquad d\hat{z}_t = -\hat{\kappa}_t \hat{z}_t dt + \hat{\sigma}_t' d\widetilde{W}_t \\
&\frac{dx_t}{x_t} = (r_t - \hat{r}_t)dt + \delta_t' d\widetilde{W}_t
\end{aligned} \tag{479}
$$

9.2.4.1. *Zero Coupon Bonds*

The domestic zero coupon bond must be under HJM framework and is given by (232)

$$
P_{t,T} = \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(b_{u,T}^2 - b_{u,t}^2\right)du - \int_s^t \left(b_{u,T} - b_{u,t}\right)d\widetilde{W}_u\right) \tag{480}
$$

$$= \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(B_{u,T}^2 - B_{u,t}^2\right)\sigma_u^2 du - B_{t,T} z_t\right)$$

The foreign zero coupon bond can also be expressed under $\mathbb{Q}$ by (473)

$$\hat{P}_{t,T} = \frac{\hat{P}_{s,T}}{\hat{P}_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(\hat{b}_{u,T}^2 - \hat{b}_{u,t}^2\right) du - \int_s^t \left(\hat{b}_{u,T} - \hat{b}_{u,t}\right) d\widehat{\widetilde{W}}_u\right)$$

$$= \frac{\hat{P}_{s,T}}{\hat{P}_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(\hat{b}_{u,T}^2 - \hat{b}_{u,t}^2\right) du + \int_s^t \delta_u' \left(\hat{b}_{u,T} - \hat{b}_{u,t}\right) du - \int_s^t \left(\hat{b}_{u,T} - \hat{b}_{u,t}\right) d\widetilde{W}_u\right) \tag{481}$$

$$= \frac{\hat{P}_{s,T}}{\hat{P}_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(\hat{B}_{u,T}^2 - \hat{B}_{u,t}^2\right)\hat{\sigma}_u^2 du + \hat{B}_{t,T}\int_s^t \hat{E}_{u,t}\hat{\sigma}_u'\delta_u du - \hat{B}_{t,T}\hat{z}_t\right)$$

Given the bond prices above, their dynamics show as follows

$$\frac{dP_{t,T}}{P_{t,T}} = r_t dt - b_{t,T}' d\widetilde{W}_t, \qquad \frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} = \hat{r}_t dt - \hat{b}_{t,T}' d\widehat{\widetilde{W}}_t = \left(\hat{r}_t + \hat{b}_{t,T}'\delta_t\right) dt - \hat{b}_{t,T}' d\widetilde{W}_t \tag{482}$$

9.2.4.2. *FX Forward Rate*

We know from (468) that $x_t \hat{P}_{t,T} M_t^{-1}$ is a $\mathbb{Q}$-martingale. If we change the numeraire from M_t to $P_{t,T}$, the forward FX rate $y_{t,T} = x_t \hat{P}_{t,T} P_{t,T}^{-1}$ must be a martingale under domestic T-forward measure $\mathbb{Q}^T$, that is

$$y_{t,T} = \frac{x_t \hat{P}_{t,T}}{P_{t,T}} = \mathbb{E}_t^T\left[\frac{x_T \hat{P}_{T,T}}{P_{T,T}}\right] = \mathbb{E}_t^T[x_T] = \mathbb{E}_t^T\left[y_{T,T}\right] \tag{483}$$

Given the bond price dynamics (482), the dynamics of the forward FX rate (and its inverse) can be inferred from (468) via the change of numeraire (22) from M_t to $P_{t,T}$, that is

$$\frac{dy_{t,T}}{y_{t,T}} = \delta_{t,T}' dW_t^T, \qquad d\frac{1}{y_{t,T}} = \frac{1}{y_{t,T}}\left(\delta_{t,T}^2 dt - \delta_{t,T}' dW_t^T\right) \tag{484}$$

where the FX forward volatility $\delta_{t,T} = \delta_t - \hat{b}_{t,T} + b_{t,T}$ and

$$dW_t^T = d\widetilde{W}_t + b_{t,T} dt \tag{485}$$

The (484) shows, in short term the FX forward volatility $\delta_{t,T}$ is dominated by FX spot volatility δ_t, so the assumption of deterministic interest rates is acceptable. However, when the term gets longer, the significance of bond volatilities $\hat{b}_{t,T}$ and $b_{t,T}$ grows, and therefore rates dynamics must be accounted for long-dated FX derivatives.

The FX forward rate can then be derived by integrating (484) from s to t

$$\begin{aligned}\frac{y_{t,T}}{y_{s,T}} &= \exp\left(-\frac{1}{2}\int_s^t \delta_{u,T}^2 du + \int_s^t \delta_{u,T}' dW_u^T\right)\\ &= \exp\left(-\frac{1}{2}\int_s^t \delta_{u,T}^2 du + \int_s^t \delta_{u,T}' b_{u,T} du + \int_s^t \delta_{u,T}' d\widetilde{W}_u\right)\\ &= \exp\left(-\int_s^t \frac{\left(\delta_u - \hat{b}_{u,T}\right)^2 - b_{u,T}^2}{2} du + \int_s^t \left(\delta_u - \hat{b}_{u,T} + b_{u,T}\right)' d\widetilde{W}_u\right)\end{aligned} \tag{486}$$

Consequently, we have the FX spot $x_t = y_{t,t}$

$$\begin{aligned}x_t &= y_{s,t} \exp\left(-\frac{1}{2}\int_s^t \left(\delta_u - \hat{b}_{u,t} + b_{u,t}\right)^2 du + \int_s^t \left(\delta_u - \hat{b}_{u,t} + b_{u,t}\right)' W_u^t\right)\\ &= x_s \frac{\hat{P}_{s,t}}{P_{s,t}} \exp\left(-\int_s^t \frac{\left(\delta_u - \hat{b}_{u,t}\right)^2 - b_{u,t}^2}{2} du + \int_s^t \left(\delta_u - \hat{b}_{u,t} + b_{u,t}\right)' d\widetilde{W}_u\right)\end{aligned} \tag{487}$$

9.2.4.3. *FX Forward Rate Ratio*

Let us define a ratio between two FX forward rates (or between two forward inflation index levels) maturing at T and V respectively as

$$R_{t,T,V} = \frac{y_{t,V}}{y_{t,T}} = \frac{\hat{P}_{t,V} P_{t,T}}{\hat{P}_{t,T} P_{t,V}} \quad \forall\, t \le T \le V \tag{488}$$

its dynamics can be derived by Ito’s lemma using (484)

$$\begin{aligned}\frac{dR_{t,T,V}}{R_{t,T,V}} &= \delta_{t,V}' dW_t^V + \delta_{t,T}^2 dt - \delta_{t,T}' dW_t^T - \delta_{t,T}' \delta_{t,V} dt\\ &= \delta_{t,V}' dW_t^V + \delta_{t,T}^2 dt - \delta_{t,T}' dW_t^V + \delta_{t,T}' \left(b_{t,V} - b_{t,T}\right) dt - \delta_{t,T}' \delta_{t,V} dt\\ &= \delta_{t,T}' \left(b_{t,V} - b_{t,T} - \delta_{t,V} + \delta_{t,T}\right) dt + \left(\delta_{t,V} - \delta_{t,T}\right)' dW_t^V\\ &= \delta_{t,T}' \left(\hat{b}_{t,V} - \hat{b}_{t,T}\right) dt + \left(\delta_{t,V} - \delta_{t,T}\right)' dW_t^V\end{aligned} \tag{489}$$

$$= \left(\delta_t + b_{t,T} - \hat{b}_{t,T}\right)'\left(\hat{b}_{t,V} - \hat{b}_{t,T}\right)dt + \left(b_{t,V} - \hat{b}_{t,V} - b_{t,T} + \hat{b}_{t,T}\right)' dW_t^V$$

where we have used the expression of $\delta_{t,T}$ in (484) and the change of measure

$$\delta_{t,T} = \delta_t + b_{t,T} - \hat{b}_{t,T}, \qquad dW_t^T = dW_t^V - \left(b_{t,V} - b_{t,T}\right)dt \tag{490}$$

9.2.4.4. *Zero Coupon Swap and Year-on-Year Swap*

In the following, we will develop formulas to calculate prices of several (maybe hypothetical) derivative products on FX rate or inflation index. Let us first consider two products whose payoff depends on the FX forward rates upon maturity: zero coupon (ZC) swap and year-on-year (YoY) swap.

The ZC swap at time T swaps the fixed and floating leg as

$$\text{Fixed Leg: } (1+K)^{T-t} - 1, \qquad \text{Floating Leg: } \frac{x_T}{x_t} - 1 \tag{491}$$

The fixed leg is a simple cashflow that is easy to value. The floating leg of a ZC swap can be priced based on (483) as

$$V_{t,T}^{ZC} = \widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_T}\left(\frac{x_T}{x_t} - 1\right)\right] = P_{t,T}\mathbb{E}_t^T\left[\frac{x_T}{x_t} - 1\right] = P_{t,T}\left(\frac{y_{t,T}}{x_t} - 1\right) = \hat{P}_{t,T} - P_{t,T} \tag{492}$$

(In fact, the real zero bond $\hat{P}_{t,T}$ in inflation market is not directly observable, however we can use the relationship in (492) to construct a term structure of zero rate, given that the term structure of nominal zero bond $P_{t,T}$ and the ZC swaps are readily available [54].)

The YoY swap, on the other hand, at each time T_i swaps the fixed and floating leg as

$$\text{Fixed Leg: } K\tau_i, \qquad \text{Floating Leg: } \left(\frac{x_i}{x_{i-1}} - 1\right)\tau_i \tag{493}$$

where K is the fixed coupon rate and $\tau_i = T_i - T_{i-1}$ is the year fraction between the two dates. The i-th period of floating leg of a YoY swap can be priced as

$$V_{t,i}^{YoY} = \tau_i\widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_i}\left(\frac{x_i}{x_{i-1}} - 1\right)\right] = \tau_i P_{t,i}\left(\mathbb{E}_t^i\left[\frac{x_i}{x_{i-1}}\right] - 1\right) \tag{494}$$

The expectation of the FX rate ratio can be expressed as an expectation of the ratio of two forward bonds for (483), that is (for $s \le t < T < V$)

$$\mathbb{E}_t^V\left[\frac{x_V}{x_T}\right]=\mathbb{E}_t^V\left[\mathbb{E}_T^V\left[\frac{x_V}{x_T}\right]\right]=\mathbb{E}_t^V\left[\frac{y_{T,V}}{x_T}\right]=\mathbb{E}_t^V\left[\frac{\hat{P}_{T,V}}{P_{T,V}}\right] \tag{495}$$

or more conveniently as an expectation of the FX forward rate ratio $R_{t,T,V}$

$$\mathbb{E}_t^V\left[\frac{x_V}{x_T}\right]=\mathbb{E}_t^V\left[\frac{y_{T,V}}{y_{T,T}}\right]=\mathbb{E}_t^V\left[R_{T,T,V}\right]=R_{t,T,V}\exp\left(C_{t,T,V}\right),\qquad C_{t,T,V}=\int_t^T\delta'_{u,T}\left(\hat{b}_{u,V}-\hat{b}_{u,T}\right)du \tag{496}$$

The last equality in (496) comes from the fact that the $R_{t,T,V}$ follows a lognormal process defined in (489).

9.2.4.5. *European Option*

A European option on FX rate (or inflation index level) expiring at time T with a strike K can be priced as

$$V_{t,T}^{FX}=\widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_T}(\omega x_T-\omega K)^+\right]=P_{t,T}\mathbb{E}_t^T\left[\left(\omega y_{T,T}-\omega K\right)^+\right]=P_{t,T}\mathfrak{B}\left(K,m_y,v_y,\omega\right) \tag{497}$$

where $\omega=\pm 1$ flags a call or a put. Since $y_{t,T}$ is a martingale under domestic T-forward measure, the price be calculated by Black formula (81) with mean $m_y=y_{t,T}$ and total variance

$$\begin{aligned}v_y&=\int_s^t\delta_{u,T}^2du=\int_s^t\left(\delta_u+b_{u,T}-\hat{b}_{u,T}\right)^2du\\&=\int_s^t\left(\delta_u^2+B_{u,T}^2\sigma_u^2+\hat{B}_{u,T}^2\hat{\sigma}_u^2+2B_{u,T}\delta'_u\sigma_u-2\hat{B}_{u,T}\delta'_u\hat{\sigma}_u-2B_{u,T}\hat{B}_{u,T}\sigma'_u\hat{\sigma}_u\right)du\end{aligned} \tag{498}$$

9.2.4.6. *Forward Start Option*

Cliquet option (or ratchet option) is a portfolio of forward start options that periodically settles and resets its strike price at the level of the underlying during the time of settlement. Each forward start option comprising the cliquet enters into force when the previous option expires. For example, a cliquet option on the FX rate ratio (or inflation index level ratio) with a strike K can be priced as (for $s\le t=T_0<\cdots<T_n=T$)

$$V_{t,T}^{CO}=\widetilde{\mathbb{E}}_t\left[\sum_{i=1}^n\frac{M_t}{M_i}\left(\omega\frac{x_i}{x_{i-1}}-\omega K\right)^+\right]=\sum_{i=1}^n\widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_i}\left(\omega\frac{x_i}{x_{i-1}}-\omega K\right)^+\right]=\sum_{i=1}^nV_{t,i-1,i}^{FS} \tag{499}$$

where $\omega = \pm 1$ flags a call or a put and $V^{FS}_{t,i-1,i}$ is the present value of the i-th forward start option. A forward start option can be priced by (for $s \le t < T < V$)

$$V^{FS}_{t,T,V} = \widetilde{\mathbb{E}}_t\left[\frac{M_t}{M_V}\left(\omega\frac{x_V}{x_T} - \omega K\right)^+\right] = P_{t,V}\mathbb{E}^V_t\left[\left(\omega R_{V,T,V} - \omega K\right)^+\right] = P_{t,V}\mathfrak{B}(K, m_R, v_R, \omega) \tag{500}$$

Since the $R_{t,T,V}$ is a lognormal process under domestic V-forward measure $\mathbb{Q}^V$, the above expectation can be calculated by Black formula (81) with mean m_R and total variance v_R. The mean has been given in (496) while the total variance can be computed as

$$v_R = \widehat{\mathbb{V}}^V_t[\log R_{V,T,V}] = \int_t^V \left(\delta_{u,V} - \delta_{u,T}\right)^2 du = \int_t^T \left(\delta_{u,V} - \delta_{u,T}\right)^2 du + \int_T^V \delta^2_{u,V} du \tag{501}$$

In (501), we decomposes the integral into two pieces. The first integral comes from the randomness of the forward rate ratio which has both numerator and denominator active until T, whereas the second comes solely from the randomness of the numerator because the denominator has been fixed at T.

In the Hull-White model, the dynamics of forward FX rate ratio in (489) becomes

$$\frac{dR_{t,T,V}}{R_{t,T,V}} = \hat{B}_{T,V}\hat{E}_{t,T}\hat{\sigma}'_t\left(\delta_t + B_{t,T}\sigma_t - \hat{B}_{t,T}\hat{\sigma}_t\right)dt + \left(B_{T,V}E_{t,T}\sigma_t - \hat{B}_{T,V}\hat{E}_{t,T}\hat{\sigma}_t\right)' dW^V_t \tag{502}$$

If we further assume time-invariant $\kappa, \hat{\kappa}, \sigma$ and $\hat{\sigma}$ for rates and δ for FX spot, the mean m_R in (496) and total variance v_R in (501) of the forward FX rate ratio can be derived analytically. The mean [55]can be computed as

$$\mathbb{E}^V_t\left[R_{V,T,V}\right] = R_{t,T,V}\exp\left(C_{t,T,V}\right) = \frac{\hat{P}_{t,V}P_{t,T}}{\hat{P}_{t,T}P_{t,V}}\exp\left(C_{t,T,V}\right) \qquad \text{and}$$

$$\begin{aligned} C_{t,T,V} &= \int_t^T \hat{B}_{T,V}\hat{E}_{u,T}\hat{\sigma}'_u\left(\delta_u + B_{u,T}\sigma_u - \hat{B}_{u,T}\hat{\sigma}_u\right)du \\ &= \hat{B}_{T,V}\hat{\sigma}'\left(\hat{B}_{t,T}\delta - \frac{1}{2}\hat{B}^2_{t,T}\hat{\sigma} + \frac{\hat{B}_{t,T} + \hat{\kappa}\hat{B}_{t,T}B_{t,T} - B_{t,T}}{\kappa + \hat{\kappa}}\sigma\right) \end{aligned} \tag{503}$$

where we have for constant κ and $\hat{\kappa}$

$$B_{t,T}=\frac{1-e^{-\kappa(T-t)}}{\kappa},\qquad \hat{B}_{t,T}=\frac{1-e^{-\hat{\kappa}(T-t)}}{\hat{\kappa}} \tag{504}$$

and the equation

$$\int_t^T \hat{E}_{u,T}B_{u,T}du=\int_t^T \hat{E}_{u,T}\frac{1-E_{u,T}}{\kappa}du=\frac{\hat{B}_{t,T}}{\kappa}-\frac{1-e^{-(\kappa+\hat{\kappa})(T-t)}}{\kappa(\kappa+\hat{\kappa})}=\frac{\hat{B}_{t,T}+\hat{\kappa}\hat{B}_{t,T}B_{t,T}-B_{t,T}}{\kappa+\hat{\kappa}} \tag{505}$$

The total variance [56]can be computed as

$$\mathbb{V}_t^V\left[\log R_{V,T,V}\right]=\int_t^T\left(B_{T,V}E_{u,T}\sigma_u-\hat{B}_{T,V}\hat{E}_{u,T}\hat{\sigma}_u\right)^2du+\int_T^V\left(\delta_u-\hat{B}_{u,V}\hat{\sigma}_u+B_{u,V}\sigma_u\right)^2du \tag{506}$$

We derive the first integral as

$$\begin{aligned}&\int_t^T\left(B_{T,V}E_{u,T}\sigma_u-\hat{B}_{T,V}\hat{E}_{u,T}\hat{\sigma}_u\right)^2du\\&=B_{T,V}^2\sigma^2\int_t^T E_{u,T}^2du+\hat{B}_{T,V}^2\hat{\sigma}^2\int_t^T\hat{E}_{u,T}^2du-2B_{T,V}\hat{B}_{T,V}\sigma'\hat{\sigma}\int_t^T E_{u,T}\hat{E}_{u,T}du\\&=B_{T,V}^2\mathcal{B}_{2\kappa;t,T}\sigma^2+\hat{B}_{T,V}^2\mathcal{B}_{2\hat{\kappa};t,T}\hat{\sigma}^2-2B_{T,V}\hat{B}_{T,V}\mathcal{B}_{\kappa+\hat{\kappa};t,T}\sigma'\hat{\sigma}\end{aligned} \tag{507}$$

and the second integral as

$$\begin{aligned}&\int_T^V\left(\delta_u-\hat{B}_{u,V}\hat{\sigma}_u+B_{u,V}\sigma_u\right)^2du\\&=\int_T^V\left(\delta_u^2+\hat{B}_{u,V}^2\hat{\sigma}_u^2+B_{u,V}^2\sigma_u^2-2\hat{B}_{u,V}\delta_u'\hat{\sigma}_u+2B_{u,V}\delta_u'\sigma_u-2B_{u,V}\hat{B}_{u,V}\sigma_u'\hat{\sigma}_u\right)du\\&=\tau\delta^2+\frac{\tau-2\hat{B}_{T,V}+\mathcal{B}_{2\hat{\kappa};T,V}}{\hat{\kappa}^2}\hat{\sigma}^2+\frac{\tau-2B_{T,V}+\mathcal{B}_{2\kappa;T,V}}{\kappa^2}\sigma^2-2\frac{\tau-\hat{B}_{T,V}}{\hat{\kappa}}\delta'\hat{\sigma}+2\frac{\tau-B_{T,V}}{\kappa}\delta'\sigma\\&\qquad-2\frac{\tau-B_{T,V}-\hat{B}_{T,V}+\mathcal{B}_{\kappa+\hat{\kappa};T,V}}{\kappa\hat{\kappa}}\sigma'\hat{\sigma}\end{aligned} \tag{508}$$

where we define $\tau=V-T$ and function

$$\mathcal{B}_{\omega;t,T}=\frac{1-e^{-\omega(T-t)}}{\omega} \tag{509}$$

9.3. Three-factor Model: Modeling Rate Spread

We discuss here a variant of the previous 3-factor model. The only modification we have made is that in this model the spread between the domestic and foreign forward rates $\bar{f}_{t,T} = f_{t,T} - \hat{f}_{t,T}$ is modeled (an accent "bar" denotes a quantity related to rate spread), rather than the foreign forward rate $\hat{f}_{t,T}$ itself.

9.3.1. Model Definition

In this model, the domestic forward rate $f_{t,T}$, the forward rate spread $\bar{f}_{t,T}$ and the FX spot rate X_t are described by the following SDE under physical measure $\mathbb{P}$

$$df_{t,T} = \alpha_{t,T}dt + \beta'_{t,T}dW_t, \qquad d\bar{f}_{t,T} = \bar{\alpha}_{t,T}dt + \bar{\beta}'_{t,T}dW_t, \qquad \frac{dx_t}{x_t} = \mu_t dt + \delta'_t dW_t \tag{510}$$

where we will use, once again, market tradable assets to identify the market price of risk vector λ_t for changing the probability measure by (462) from $\mathbb{P}$ to the $\mathbb{Q}$.

9.3.2. Domestic Risk Neutral Measure

We use the aforementioned three market tradable assets to identify the λ_t: 1) domestic bond $P_{t,T}$, 2) foreign money market account $x_t\widehat{M}_t$ and 3) the foreign bond $x_t\hat{P}_{t,T}$

$$\begin{aligned} &P_{t,T} = \exp\left(-\int_t^T f_{t,u}du\right), \qquad x_t\widehat{M}_t = x_t \exp\left(\int_s^t (r_u - \bar{r}_t)du\right) \\ &x_t\hat{P}_{t,T} = x_t \exp\left(-\int_t^T \left(f_{t,u} - \bar{f}_{t,u}\right)du\right) \end{aligned} \tag{511}$$

where the spot spread $\bar{r}_t = r_t - \hat{r}_t$ is the difference between domestic and foreign short rate (or called inflation short rate in inflation markets).

The first two assets are simple to handle. We can follow the same derivation for (465) and (467) to obtain two equations that hold for λ_t

$$\beta'_{t,T}\lambda_t = \alpha_{t,T} - \beta'_{t,T}b_{t,T} \qquad \text{and} \qquad \delta'_t\lambda_t = \mu_t - \bar{r}_t \tag{512}$$

where as usual we define

$$a_{t,T} = \int_t^T \alpha_{t,u}du\,, \qquad b_{t,T} = \int_t^T \beta_{t,u}du\,, \qquad \bar{a}_{t,T} = \int_t^T \bar{\alpha}_{t,u}du\,, \qquad \bar{b}_{t,T} = \int_t^T \bar{\beta}_{t,u}du \tag{513}$$

The last asset $x_t\hat{P}_{t,T}$ determines the third equation. At first, we derive the dynamics of $\hat{P}_{t,T}$

$$\frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} = \left(r_t - \bar{r}_t + \bar{a}_{t,T} - a_{t,T} + \frac{1}{2}\left(\bar{b}_{t,T} - b_{t,T}\right)^2\right)dt + \left(\bar{b}_{t,T} - b_{t,T}\right)' dW_t \tag{514}$$

The $x_t \hat{P}_{t,T} M_t^{-1}$ is a $\mathbb{Q}$-martingale, whose dynamics can be derived as

$$\begin{aligned}
\frac{d\left(x_t \hat{P}_{t,T} M_t^{-1}\right)}{x_t \hat{P}_{t,T} M_t^{-1}} &= \frac{dx_t}{x_t} + \frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} + \frac{dM_t^{-1}}{M_t^{-1}} + \frac{dx_t}{x_t}\frac{d\hat{P}_{t,T}}{\hat{P}_{t,T}} \\
&= \left(\mu_t + r_t - \bar{r}_t + \bar{a}_{t,T} - a_{t,T} + \frac{1}{2}\left(\bar{b}_{t,T} - b_{t,T}\right)^2 - r_t + \delta_t'\left(\bar{b}_{t,T} - b_{t,T}\right)\right)dt \\
&\qquad + \left(\delta_t + \bar{b}_{t,T} - b_{t,T}\right)' dW_t \\
&= \left(\delta_t'\lambda_t + \bar{a}_{t,T} - a_{t,T} + \left(\frac{\bar{b}_{t,T} - b_{t,T}}{2} + \delta_t\right)'\left(\bar{b}_{t,T} - b_{t,T}\right)\right)dt + \left(\delta_t + \bar{b}_{t,T} - b_{t,T}\right)' dW_t \\
&= \left(\bar{a}_{t,T} - a_{t,T} + \left(\frac{\bar{b}_{t,T} - b_{t,T}}{2} + \delta_t - \lambda_t\right)'\left(\bar{b}_{t,T} - b_{t,T}\right)\right)dt + \left(\delta_t + \bar{b}_{t,T} - b_{t,T}\right)' d\widetilde{W}_t
\end{aligned} \tag{515}$$

where we have used the second identity in (512) and the (462) to reach the last equality. The drift term must vanish, so we have

$$\bar{a}_{t,T} - a_{t,T} + \left(\frac{\bar{b}_{t,T} - b_{t,T}}{2} + \delta_t - \lambda_t\right)'\left(\bar{b}_{t,T} - b_{t,T}\right) = 0 \tag{516}$$

By taking $\frac{\partial}{\partial T}$ and plugging in the first identity in (512), we obtain

$$\begin{aligned}
&\bar{\alpha}_{t,T} - \alpha_{t,T} + \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)'\left(\bar{b}_{t,T} - b_{t,T} + \delta_t - \lambda_t\right) = 0 \\
&\Longrightarrow \bar{\alpha}_{t,T} - \alpha_{t,T} + \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)'\left(\bar{b}_{t,T} - b_{t,T} + \delta_t\right) - \bar{\beta}_{t,T}'\lambda_t + \beta_{t,T}'\lambda_t = 0 \\
&\Longrightarrow \bar{\beta}_{t,T}'\lambda_t = \bar{\alpha}_{t,T} + \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)'\left(\bar{b}_{t,T} - b_{t,T} + \delta_t\right) - \beta_{t,T}' b_{t,T} \\
&\Longrightarrow \bar{\beta}_{t,T}'\lambda_t = \bar{\alpha}_{t,T} + \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)'\left(\bar{b}_{t,T} + \delta_t\right) - \bar{\beta}_{t,T}' b_{t,T}
\end{aligned} \tag{517}$$

Now we summarize as follows the three equations that determine the λ_t

$$\begin{aligned}
&\beta_{t,T}'\lambda_t = \alpha_{t,T} - \beta_{t,T}' b_{t,T} \\
&\bar{\beta}_{t,T}'\lambda_t = \bar{\alpha}_{t,T} + \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)'\left(\bar{b}_{t,T} + \delta_t\right) - \bar{\beta}_{t,T}' b_{t,T}
\end{aligned} \tag{518}$$

$$\delta_t'\lambda_t = \mu_t - \bar{r}_t$$

After changing measure from $\mathbb{P}$ to $\mathbb{Q}$ by (462), the model in (510) becomes [57]

$$\begin{aligned} df_{t,T} &= \beta_{t,T}' b_{t,T} dt + \beta_{t,T}' d\widetilde{W}_t \\ d\bar{f}_{t,T} &= \left(\bar{\beta}_{t,T}' b_{t,T} - \left(\bar{\beta}_{t,T} - \beta_{t,T}\right)' \left(\bar{b}_{t,T} + \delta_t\right)\right) dt + \bar{\beta}_{t,T}' d\widetilde{W}_t \\ \frac{dx_t}{x_t} &= \bar{r}_t dt + \delta_t' d\widetilde{W}_t \end{aligned} \tag{519}$$

9.3.3. The Hull-White Model

Again, we assume the domestic spot rate r_t and the spread $\bar{r}_t$ follow the Hull-White model where the volatilities of forward rates have the form

$$\begin{aligned} &\beta_{t,T} = E_{t,T}\sigma_t, \qquad b_{t,T} = B_{t,T}\sigma_t, \qquad E_{t,T} = \exp\left(-\int_t^T \kappa_u du\right), \qquad B_{t,T} = \int_t^T E_{t,u} du \\ &\bar{\beta}_{t,T} = \bar{E}_{t,T}\bar{\sigma}_t, \qquad \bar{b}_{t,T} = \bar{B}_{t,T}\bar{\sigma}_t, \qquad \bar{E}_{t,T} = \exp\left(-\int_t^T \bar{\kappa}_u du\right), \qquad \bar{B}_{t,T} = \int_t^T \bar{E}_{t,u} du \end{aligned} \tag{520}$$

The short rate r_t can be derived from integrating the forward rate equation for $f_{t,T}$ in (519) from start time s to t, that is

$$f_{t,T} = f_{s,T} + \int_s^t \beta_{u,T}' b_{u,T} du + \int_s^t \beta_{u,T}' d\widetilde{W}_u \tag{521}$$

Taking $T = t$ yields

$$r_t = f_{s,t} + \int_s^t \beta_{u,t}' b_{u,t} du + z_t, \qquad z_t = \int_s^t \beta_{u,t}' d\widetilde{W}_u\,, \qquad dz_t = -\kappa_t z_t dt + \sigma_t' d\widetilde{W}_t \tag{522}$$

Similarly, we can write $\bar{r}_t$, the rate spread (or inflation short rate), by integrating the forward rate equation for $\bar{f}_{t,T}$ in (519) from s to t, that is

$$\bar{f}_{t,T} = \bar{f}_{s,T} + \int_s^t \left(\bar{\beta}_{u,T}' b_{u,T} - \left(\bar{\beta}_{u,T} - \beta_{u,T}\right)' \left(\bar{b}_{u,T} + \delta_u\right)\right) du + \int_s^t \bar{\beta}_{u,T}' d\widetilde{W}_u \tag{523}$$

Taking $T = t$ yields [58]

$$\begin{aligned} &\bar{r}_t = \bar{f}_{s,t} + \int_s^t \left(\bar{\beta}'_{u,t} b_{u,t} - \left(\bar{\beta}_{u,t} - \beta_{u,t}\right)' \left(\bar{b}_{u,t} + \delta_u\right)\right) du + \bar{z}_t \\ &\bar{z}_t = \int_s^t \bar{\beta}'_{u,t} d\widetilde{W}_u, \qquad d\bar{z}_t = -\bar{\kappa}_t \bar{z}_t dt + \bar{\sigma}'_t d\widetilde{W}_t \end{aligned} \tag{524}$$

Based on the above results, we can write the model in terms of short rate

$$\begin{aligned} &r_t = f_{s,t} + \int_s^t \beta'_{u,t} b_{u,t} du + z_t \\ &dz_t = -\kappa_t z_t dt + \sigma'_t d\widetilde{W}_t \\ &\bar{r}_t = \bar{f}_{s,t} + \int_s^t \left(\bar{\beta}'_{u,t} b_{u,t} - \left(\bar{\beta}_{u,t} - \beta_{u,t}\right)' \left(\bar{b}_{u,t} + \delta_u\right)\right) du + \bar{z}_t \\ &d\bar{z}_t = -\bar{\kappa}_t \bar{z}_t dt + \bar{\sigma}'_t d\widetilde{W}_t \\ &\frac{dx_t}{x_t} = \bar{r}_t dt + \delta'_t d\widetilde{W}_t \end{aligned} \tag{525}$$

9.3.3.1. *Zero Coupon Bonds*

The domestic zero-coupon bond is given in (292)

$$\begin{aligned} P_{t,T} &= \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(b_{u,T}^2 - b_{u,t}^2\right) du - \int_s^t \left(b_{u,T} - b_{u,t}\right) d\widetilde{W}_u\right) \\ &= \frac{P_{s,T}}{P_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(B_{u,T}^2 - B_{u,t}^2\right)\sigma_u^2 du - B_{t,T} z_t\right) \end{aligned} \tag{526}$$

The foreign zero-coupon bond can be derived using (521) and (523)

$$\begin{aligned} \hat{P}_{t,T} &= \exp\left(-\int_t^T \left(f_{t,v} - \bar{f}_{t,v}\right) dv\right) = P_{t,T} \exp\left(\int_t^T \bar{f}_{t,v} dv\right) \\ &= P_{t,T} \exp\left(\int_t^T \left(\bar{f}_{s,v} + \int_s^t \left(\bar{\beta}'_{u,v} b_{u,v} - \left(\bar{\beta}_{u,v} - \beta_{u,v}\right)' \left(\bar{b}_{u,v} + \delta_u\right)\right) du + \int_s^t \bar{\beta}'_{u,v} d\widetilde{W}_u\right) dv\right) \\ &= P_{t,T} \frac{\hat{P}_{s,t,T}}{P_{s,t,T}} \exp\left(\int_s^t \int_t^T \left(\bar{\beta}'_{u,v} b_{u,v} - \left(\bar{\beta}_{u,v} - \beta_{u,v}\right)' \left(\bar{b}_{u,v} + \delta_u\right)\right) dv\, du + \int_s^t \int_t^T \bar{\beta}'_{u,v} dv\, d\widetilde{W}_u\right) \end{aligned} \tag{527}$$

$$= P_{t,T}\frac{\hat{P}_{s,t,T}}{P_{s,t,T}}\exp\left(\int_s^t\left(\bar{b}'_{u,T}b_{u,T}-\bar{b}'_{u,t}b_{u,t}-\frac{\bar{b}^2_{u,T}-\bar{b}^2_{u,t}}{2}+\delta'_u\left(b_{u,T}-b_{u,t}-\bar{b}_{u,T}+\bar{b}_{u,t}\right)\right)du\right.$$
$$\left.+\int_s^t\left(\bar{b}_{u,T}-\bar{b}_{u,t}\right)'d\widetilde{W}_u\right)$$
$$= P_{t,T}\frac{\hat{P}_{s,t,T}}{P_{s,t,T}}\exp\left(\int_s^t\left(\left(\bar{B}_{u,T}B_{u,T}-\bar{B}_{u,t}B_{u,t}\right)\bar{\sigma}'_u\sigma_u-\frac{\bar{B}^2_{u,T}-\bar{B}^2_{u,t}}{2}\bar{\sigma}^2_u\right.\right.$$
$$\left.\left.+\delta'_u\left(B_{t,T}E_{u,t}\sigma_u-\bar{B}_{t,T}\bar{E}_{u,t}\bar{\sigma}_u\right)\right)du+\bar{B}_{t,T}\bar{z}_t\right)$$

9.3.3.2. *FX Forward Rate*

Since the FX forward rate $y_{t,T}=\frac{x_t\hat{P}_{t,T}}{P_{t,T}}$ is a martingale under $\mathbb{Q}^T$ as shown in (483), its dynamics can be assumed to be in the same form of (484) but with a different specification of $\delta_{t,T}$. Given the result in (515), the $\delta_{t,T}$ can be easily obtained through the change of numeraire technique (22), where the ex-numeraire M_t has nil volatility and the new numeraire $P_{t,T}$ has a volatility of $-b_{t,T}$. Therefore, we have the FX forward rate dynamics

$$\frac{dy_{t,T}}{y_{t,T}}=\delta'_{t,T}dW^T_t,\qquad d\frac{1}{y_{t,T}}=\frac{1}{y_{t,T}}\left(\delta^2_{t,T}dt-\delta'_{t,T}dW^T_t\right) \tag{528}$$

with FX forward volatility $\delta_{t,T}=\delta_t+\bar{b}_{t,T}$ and $dW^T_t=d\widetilde{W}_t+b_{t,T}dt$.

The forward FX rate can then be derived by integrating (484) from s to t

$$\begin{aligned}\frac{y_{t,T}}{y_{s,T}}&=\exp\left(-\frac{1}{2}\int_s^t\delta^2_{u,T}du+\int_s^t\delta'_{u,T}dW^T_u\right)\\&=\exp\left(-\frac{1}{2}\int_s^t\delta^2_{u,T}du+\int_s^t\delta'_{u,T}b_{u,T}du+\int_s^t\delta'_{u,T}d\widetilde{W}_u\right)\\&=\exp\left(-\frac{1}{2}\int_s^t\left(\delta_u+\bar{b}_{u,T}\right)'\left(\delta_u+\bar{b}_{u,T}-2b_{u,T}\right)du+\int_s^t\left(\delta_u+\bar{b}_{u,T}\right)'d\widetilde{W}_u\right)\end{aligned} \tag{529}$$

Consequently, we have the FX spot $x_t=y_{t,t}$

$$\begin{aligned} x_t &= y_{s,t} \exp\left(-\frac{1}{2}\int_s^t \left(\delta_u + \bar{b}_{u,t}\right)^2 du + \int_s^T \left(\delta_u + \bar{b}_{u,t}\right)' W_u^t\right) \\ &= x_s \frac{\hat{P}_{s,t}}{P_{s,t}} \exp\left(-\frac{1}{2}\int_s^t \left(\delta_u + \bar{b}_{u,t}\right)'\left(\delta_u + \bar{b}_{u,t} - 2b_{u,t}\right) du + \int_s^T \left(\delta_u + \bar{b}_{u,t}\right)' d\widetilde{W}_u\right) \end{aligned} \tag{530}$$

9.3.3.3. *FX Forward Rate Ratio*

The dynamics of FX forward rate ratio defined in (488) can be derived from (489) with FX forward volatility $\delta_{t,T} = \delta_t + \bar{b}_{t,T}$

$$\begin{aligned} \frac{dR_{t,T,V}}{R_{t,T,V}} &= \delta'_{t,V} dW_t^V + \delta_{t,T}^2 dt - \delta'_{t,T} dW_t^T - \delta'_{t,T}\delta_{t,V} dt \\ &= \delta'_{t,V} dW_t^V + \delta_{t,T}^2 dt - \delta'_{t,T} dW_t^V + \delta'_{t,T}\left(b_{t,V} - b_{t,T}\right) dt - \delta'_{t,T}\delta_{t,V} dt \\ &= \delta'_{t,T}\left(b_{t,V} - b_{t,T} - \delta_{t,V} + \delta_{t,T}\right) dt + \left(\delta_{t,V} - \delta_{t,T}\right)' dW_t^V \\ &= \left(\delta_t + \bar{b}_{t,T}\right)'\left(b_{t,V} - b_{t,T} - \bar{b}_{t,V} + \bar{b}_{t,T}\right) dt + \left(\bar{b}_{t,V} - \bar{b}_{t,T}\right)' dW_t^V \end{aligned} \tag{531}$$

9.3.3.4. *Zero Coupon Swap and Year-on-Year Swap*

Following our previous discussion, the floating leg of a ZC swap can be priced using the same formula in (492). The i-th period of floating leg of a YoY swap can also be priced using (494) with the expectation of FX rate ratio calculated as

$$\mathbb{E}_t^V\left[\frac{x_V}{x_T}\right] = R_{t,T,V} \exp\left(C_{t,T,V}\right), \qquad C_{t,T,V} = \int_t^T \delta'_{u,T}\left(b_{u,V} - b_{u,T} - \bar{b}_{u,V} + \bar{b}_{u,T}\right) du \tag{532}$$

9.3.3.5. *European Option*

A European option on FX rate (or inflation index level) can be priced by (497) with mean $m_y = y_{t,T}$ and total variance

$$v_y = \int_s^t \delta_{u,T}^2 du = \int_s^t \left(\delta_u + \bar{B}_{u,T}\bar{\sigma}_u\right)^2 du \tag{533}$$

9.3.3.6. *Forward Start Option*

The forward start option, as aforementioned, can be priced by (500) with mean m_R given in (532) and total variance v_R computed as

$$v_R = \int_t^T \left(\delta_{u,V} - \delta_{u,T}\right)^2 du + \int_T^V \delta_{u,V}^2 du = \int_t^T \left(\bar{b}_{u,V} - \bar{b}_{u,T}\right)^2 du + \int_T^V \left(\delta_u + \bar{b}_{u,V}\right)^2 du \tag{534}$$

10. CVA AND JOINT SIMULATION OF RATES, FX AND EQUITY

In this chapter, we will extend our previous 3-factor model and construct a hybrid model for joint simulation of interest rates, FX rates and equities across different economies. One of the applications of the model is to evaluate Counterparty Value Adjustment (CVA), the market value of counterparty credit risk.

Simply put, for example, unilateral CVA is the risk-neutral expectation of the positive part of the price distribution contingent on the counterparty default

$$\frac{CVA_{t,T}}{M_t} = \widetilde{\mathbb{E}}_t\left[\int_t^T (1-R_u)\frac{V_u^+}{M_u}\delta_{\tau-u}du\right] \tag{535}$$

where V_t^+ is the positive exposure of the portfolio, R_t the recovery rate, τ the random time of default of the counterparty and δ_t the Dirac delta at t. To simplify the problem, we may need to assume constant recovery rate and independence between portfolio value and counterparty default (i.e., there's no wrong-way/right-way risk). These assumptions reduce (535) into

$$CVA_{t,T} = (1-R)M_t\int_t^T \widetilde{\mathbb{E}}_t\left[\frac{V_u^+}{M_u}\right]d\mathbb{P}_t[\tau<u] = (1-R)\int_t^T P_{t,T}\mathbb{E}_t^T\left[\frac{V_u^+}{P_{u,T}}\right]d\mathbb{P}_t[\tau<u] \tag{536}$$

where $\mathbb{P}_t[\tau<u]$ is the cumulative default probability function implied from counterparty's CDS spreads at time t. Monte Carlo simulation is often employed to evaluate the present value of the positive exposure. The simulation can be performed under either (domestic) risk neutral measure or T-forward measure. Whether one measure is superior to the other depends on the complexity of the associated numeraire to be evaluated. When the interest rate is modeled by an affine term structure model, the T-forward measure is more advantageous in simulation based methods, because its associated numeraire $P_{t,T}$ is analytically tractable. Based on our previous derivation (473) and (485), the change of measure is straightforward and can be done by the following formulas

$$d\widetilde{W}_t = dW_t^T - b_{t,T}dt, \qquad d\widehat{\widetilde{W}}_t = d\widetilde{W}_t - \delta_t dt = dW_t^T - \left(\delta_t + b_{t,T}\right)dt \tag{537}$$

where $\widetilde{W}_t$ and $\widehat{\widetilde{W}}_t$ are Brownian motions under domestic and foreign risk neutral measure respectively, W_t^T the Brownian motion under domestic T-forward measure, δ_t and $-b_{t,T}$ are volatility of FX spot x_t and bond $P_{t,T}$ respectively.

10.1. Modeling Risk Factors

Here we consider three types of risk factors to be modeled: interest rates, FX rates and equities, in both domestic and foreign economy. These risk factors are simulated under a unified probability measure: the domestic T-forward measure, providing its simplicity in evaluation of the associated numeraire.

10.1.1. Domestic Economy

Firstly, we model the interest rate and equity in domestic economy. The interest rate is modeled by a one-factor Hull-White model, such that

$$\begin{aligned}
&r_t = \phi_{s,t} + z_t, \qquad \phi_{s,t} = f_{s,t} + \int_s^t \beta'_{u,t} b_{u,t} du \\
&z_t = \int_s^t \beta'_{u,t} d\widetilde{W}_u = -\int_s^t \beta'_{u,t} b_{u,T} du + \int_s^t \beta'_{u,t} dW_u^T \\
&dz_t = -\kappa_t z_t dt + \sigma'_t d\widetilde{W}_t = -\left(\kappa_t z_t + \sigma'_t b_{t,T}\right) dt + \sigma'_t dW_t^T
\end{aligned} \tag{538}$$

The z_t conditional on the z_v for $s < v < t < T$ is normally distributed and given by

$$z_t = E_{v,t} z_v + \int_v^t \beta'_{u,t} d\widetilde{W}_u = E_{v,t} z_v - \int_v^t \beta'_{u,t} b_{u,T} du + \int_v^t \beta'_{u,t} dW_u^T \tag{539}$$

The equity is modeled by a lognormal process with continuous dividend rate η_t and deterministic volatility ξ_t

$$\begin{aligned}
&\frac{dq_t}{q_t} = (r_t - \eta_t) dt + \xi'_t d\widetilde{W}_t = \left(r_t - \eta_t - \xi'_t b_{t,T}\right) dt + \xi'_t dW_t^T \\
&d\log q_t = \left(r_t - \eta_t - \frac{\xi_t^2}{2}\right) dt + \xi'_t d\widetilde{W}_t = \left(r_t - \eta_t - \xi'_t b_{t,T} - \frac{\xi_t^2}{2}\right) dt + \xi'_t dW_t^T
\end{aligned} \tag{540}$$

The logarithm of equity spot $\log q_t$ conditional on $\log q_v$ is normally distributed and given by

$$\begin{aligned}\log\frac{q_t}{q_v} &= \int_v^t \left(r_u - \eta_u - \frac{\xi_u^2}{2}\right) du + \int_v^t \xi_u' d\widetilde{W}_u \\ &= -\log P_{v,t} + \int_v^t \left(\frac{b_{u,t}^2 - \xi_u^2}{2} - \eta_u\right) du + \int_v^t \left(\xi_u + b_{u,t}\right)' d\widetilde{W}_u \\ &= -\log P_{v,t} + \int_v^t \left(\frac{b_{u,t}^2 - \xi_u^2}{2} - \eta_u - \left(\xi_u + b_{u,t}\right)' b_{u,T}\right) du + \int_v^t \left(\xi_u + b_{u,t}\right)' dW_u^T\end{aligned} \tag{541}$$

where the integral of domestic short rate is calculated as in (233)

$$\int_v^t r_u du = -\log P_{v,t} + \int_v^t \frac{b_{u,t}^2}{2} du + \int_v^t b_{u,t}' d\widetilde{W}_u \tag{542}$$

10.1.2. Foreign Economy

Secondly, we model the interest rate, FX rate and equity in foreign economy (denoted by accent "^"). The foreign short rate is also modeled by a one-factor Hull-White model, which is in a great similarity as in domestic economy. Following the same derivation, we have

$$\begin{aligned}&\hat{r}_t = \hat{\phi}_{s,t} + \hat{z}_t, \qquad \hat{\phi}_{s,t} = \hat{f}_{s,t} + \int_s^t \hat{\beta}_{u,t}' \hat{b}_{u,t} du \\ &\hat{z}_t = \int_s^t \hat{\beta}_{u,t}' d\widehat{\widetilde{W}}_u = -\int_s^t \hat{\beta}_{u,t}' \left(\delta_u + b_{u,T}\right) dt + \int_s^t \hat{\beta}_{u,t}' dW_u^T \\ &d\hat{z}_t = -\hat{\kappa}_t \hat{z}_t dt + \hat{\sigma}_t' d\widehat{\widetilde{W}}_t = -\left(\hat{\kappa}_t \hat{z}_t + \hat{\sigma}_t' \left(\delta_t + b_{t,T}\right)\right) dt + \hat{\sigma}_t' dW_t^T\end{aligned} \tag{543}$$

The $\hat{z}_t$ conditional on the $\hat{z}_v$ for $s < v < t < T$ is normally distributed and given by

$$\hat{z}_t = \hat{E}_{v,t} \hat{z}_v + \int_v^t \hat{\beta}_{u,t}' d\widehat{\widetilde{W}}_u = \hat{E}_{v,t} \hat{z}_v - \int_v^t \hat{\beta}_{u,t}' \left(\delta_u + b_{u,T}\right) du + \int_v^t \beta_{u,t}' dW_u^T \tag{544}$$

The FX rate is modeled by a lognormal process with stochastic short rates and deterministic volatility

$$\begin{aligned}&\frac{dx_t}{x_t} = (r_t - \hat{r}_t) dt + d\widetilde{W}_t = \left(r_t - \hat{r}_t - \delta_t' b_{t,T}\right) dt + \delta_t' dW_t^T \\ &d\log x_t = \left(r_t - \hat{r}_t - \frac{\delta_t^2}{2}\right) dt + \delta_t' d\widetilde{W}_t = \left(r_t - \hat{r}_t - \delta_t' b_{t,T} - \frac{\delta_t^2}{2}\right) dt + \delta_t' dW_t^T\end{aligned} \tag{545}$$

The logarithm of FX spot $\log x_t$ conditional on $\log x_v$ is normally distributed and given by

$$\begin{aligned}
\log\frac{x_t}{x_v} &= \int_v^t \left(r_u - \hat{r}_u - \frac{\delta_u^2}{2}\right)du + \int_v^t \delta_u' d\widetilde{W}_u \\
&= \log\frac{\hat{P}_{v,t}}{P_{v,t}} + \int_v^t \frac{b_{u,t}^2 - \hat{b}_{u,t}^2 - \delta_u^2}{2}du + \int_v^t b_{u,t}' d\widetilde{W}_u - \int_v^t \hat{b}_{u,t}' d\widehat{\widetilde{W}}_u + \int_v^t \delta_u' d\widetilde{W}_u \\
&= \log\frac{\hat{P}_{v,t}}{P_{v,t}} - \int_v^t \frac{\left(\delta_u - \hat{b}_{u,t}\right)^2 - b_{u,t}^2}{2}du + \int_v^t \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' d\widetilde{W}_u \qquad \text{as in (487)} \\
&= \log\frac{\hat{P}_{v,t}}{P_{v,t}} - \int_v^t \left(\frac{\left(\delta_u - \hat{b}_{u,t}\right)^2 - b_{u,t}^2}{2} + \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' b_{u,T}\right)du \\
&\qquad + \int_v^t \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' dW_u^T
\end{aligned} \tag{546}$$

Note that the FX forward rate $x_v\hat{P}_{v,t}/P_{v,t}$ becomes a martingale under the t-forward measure, that is

$$\log x_t - \log x_v \frac{\hat{P}_{v,t}}{P_{v,t}} = -\int_v^t \frac{\left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)^2}{2}du + \int_v^t \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' dW_u^t \tag{547}$$

The foreign equity is modeled by a lognormal process with continuous dividend rate $\hat{\eta}_t$ and deterministic volatility $\hat{\xi}_t$

$$\begin{aligned}
\frac{d\hat{q}_t}{\hat{q}_t} &= (\hat{r}_t - \hat{\eta}_t)dt + \hat{\xi}_t' d\widehat{\widetilde{W}}_t = \left(\hat{r}_t - \hat{\eta}_t - \hat{\xi}_t'\delta_t - \hat{\xi}_t' b_{t,T}\right)dt + \hat{\xi}_t' dW_t^T \\
d\log\hat{q}_t &= \left(\hat{r}_t - \hat{\eta}_t - \frac{1}{2}\hat{\xi}_t^2\right)dt + \hat{\xi}_t' d\widehat{\widetilde{W}}_t = \left(\hat{r}_t - \hat{\eta}_t - \hat{\xi}_t'\delta_t - \hat{\xi}_t' b_{t,T} - \frac{1}{2}\hat{\xi}_t^2\right)dt + \hat{\xi}_t' dW_t^T
\end{aligned} \tag{548}$$

and the logarithm of equity spot $\log\hat{q}_t$ conditional on $\log\hat{q}_v$ is normally distributed and given by

$$\begin{aligned}
\log\frac{\hat{q}_t}{\hat{q}_v} &= -\log\hat{P}_{v,t} + \int_v^t \left(\frac{\hat{b}_{u,t}^2 - \hat{\xi}_u^2}{2} - \hat{\eta}_u\right)du + \int_v^t \left(\hat{\xi}_u + \hat{b}_{u,t}\right)' d\widehat{\widetilde{W}}_u \\
&= -\log\hat{P}_{v,t} + \int_v^t \left(\frac{\hat{b}_{u,t}^2 - \hat{\xi}_u^2}{2} - \hat{\eta}_u - \left(\hat{\xi}_u + \hat{b}_{u,t}\right)'\left(\delta_u + b_{u,T}\right)\right)du + \int_v^t \left(\hat{\xi}_u + \hat{b}_{u,t}\right)' dW_u^T
\end{aligned} \tag{549}$$

10.1.3. Equity Dividends

The equity process allows for a continuous dividend rate as an input. Deterministic discrete dividend payments can be approximated using spiky dividend rates. Suppose for the foreign equity[1] we have a set of discrete dividend payments $\hat{c}_1, \hat{c}_2, \cdots, \hat{c}_n$ occurring at times $s < t < T_1 < T_2 < \cdots < T_n \le T$. This stream of dividend payments can be modeled using a piecewise constant continuous dividend rate with intervals $[t, T_1 - \Delta]$, $[T_1 - \Delta, T_1]$, $[T_1, T_2 - \Delta]$, $[T_2 - \Delta, T_2]$, $\cdots$, $[T_n - \Delta, T_n]$, $[T_n, T]$. The continuous dividend rate takes constant value $\hat{\eta}_i$ when $t \in [T_i - \Delta, T_i]$ and zero otherwise. To calculate the constant vaule $\hat{\eta}_i$, we start from the well-known martingale property. Under domestic T-forward measure, the equity $\hat{q}_t$ that pays dividends continuously (for $t < \tau < T$) admits the following identity

$$\frac{x_t \hat{q}_t}{P_{t,T}} = \mathbb{E}_t^T \left[\frac{x_\tau \hat{q}_\tau \exp\left(\int_t^\tau \hat{\eta}_u du\right)}{P_{\tau,T}} \right] \tag{550}$$

The present value of the change in the equity over the course of each individual dividend payment period $[T_i - \Delta, T_i]$ with infinitesimal Δ (such that the $P_{T_i - \Delta, T}$ and $P_{T_i, T}$ differ only negligibly) must equal the present value of the corresponding dividend, that is

$$\begin{aligned} \frac{x_t \hat{c}_i \hat{P}_{t,i}}{P_{t,T}} &= \mathbb{E}_t^T \left[\frac{x_{T_i - \Delta} \hat{q}_{T_i - \Delta} - x_{T_i} \hat{q}_{T_i}}{P_{T_i,T}} \right] = \frac{x_t \hat{q}_t}{P_{t,T}} \exp\left(-\int_t^{T_i - \Delta} \hat{\eta}_u du \right) - \frac{x_t \hat{q}_t}{P_{t,T}} \exp\left(-\int_t^{T_i} \hat{\eta}_u du \right) \\ &= \frac{x_t \hat{q}_t}{P_{t,T}} \left(1 - e^{-\hat{\eta}_i \Delta}\right) \prod_{k=1}^{i-1} \left(1 - e^{-\hat{\eta}_k \Delta}\right) \end{aligned} \tag{551}$$

This translates into a recursive formula

$$\hat{\eta}_i = -\frac{1}{\Delta} \log\left(1 - \frac{\hat{c}_i \hat{P}_{t,i}}{\hat{q}_t \prod_{k=1}^{i-1} (1 - e^{-\hat{\eta}_k \Delta})} \right) \tag{552}$$

which can be used iteratively to determine the dividend rate $\hat{\eta}_i$ from a sequence of discrete dividend dates T_i and amounts $\hat{c}_i$.

10.1.4. Equity Option

[1] Without loss of generality, we derive the formulas based on a foreign equity. These formulas however can be easily adapted for a domestic equity with a constant FX rate $x_t = 1$.

Since the equity spot is lognormally distributed, the time t price of a European option maturing at T on equity $\hat{q}_T$ can be calculated by Black formula (81)

$$\hat{V}_{t,T}^{EQ} = \hat{P}_{t,T}\hat{\mathbb{E}}_t^T[(\omega\hat{q}_T - \omega K)^+] = \hat{P}_{t,T}\mathfrak{B}(K, \hat{m}, \hat{v}, \omega) \tag{553}$$

where $\omega = \pm 1$ flags a call or a put, and the mean and variance of $\hat{q}_T$ are given by

$$\hat{m} = \hat{\mathbb{E}}_t^T[\hat{q}_T] = \frac{\hat{q}_t \exp\left(-\int_t^T \hat{\eta}_u du\right)}{\hat{P}_{t,T}}, \qquad \hat{v} = \hat{\mathbb{V}}_t^T[\hat{q}_T] = \int_t^T \left(\hat{\xi}_u + \hat{b}_{u,T}\right)^2 du \tag{554}$$

Note that under foreign T-forward measure, we have the following martingale

$$\frac{\hat{q}_t}{\hat{P}_{t,T}} = \hat{\mathbb{E}}_t^T\left[\frac{\hat{q}_\tau \exp\left(\int_t^\tau \hat{\eta}_u du\right)}{\hat{P}_{\tau,T}}\right] \tag{555}$$

10.2. Monte Carlo Simulation

Monte Carlo simulation is performed under the domestic T-forward measure. The risk factors z_t, $\log q_t$, $\log x_t$, $\hat{z}_t$ and $\log \hat{q}_t$, given in (539) (541) (544) (546) and (549) respectively, follow a joint normal distribution. Providing that the instantaneous volatilities are piece-wise constant functions, there exist analytical formulas for the mean and the total variance (i.e., the integral of instantaneous covariance over one time-step) of the joint normal. However, this involves intensive algebraic derivations and complicates the implementation, especially when the number of risk factors is large. Alternatively, we seek an analytical mean while the total covariance is integrated numerically.

Writing the risk factor SDE's in a matrix form (driven by the same n-dimensional standard Brownian motion), we have

$$d\omega_t = (H_t\omega_t + h_t)dt + \Sigma_t' d\widetilde{W}_t = \left(H_t\omega_t + h_t - \Sigma_t' b_{t,T}\right)dt + \Sigma_t' dW_t^T \tag{556}$$

where the following vectors and matrices are defined

$$\underset{5\times 1}{\omega_t} = \begin{bmatrix} z_t \\ \hat{z}_t \\ \log x_t \\ \log q_t \\ \log \hat{q}_t \end{bmatrix}, \qquad \underset{5\times 5}{H_t} = \begin{bmatrix} -\kappa_t & 0 & 0 & 0 & 0 \\ 0 & -\hat{\kappa}_t & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad \underset{5\times 1}{h_t} = \begin{bmatrix} 0 \\ -\hat{\sigma}_t' \delta_t \\ \phi_{s,t} - \hat{\phi}_{s,t} - \frac{\delta_t^2}{2} \\ \phi_{s,t} - \eta_t - \frac{\xi_t^2}{2} \\ \hat{\phi}_{s,t} - \hat{\eta}_t - \hat{\xi}_t' \delta_t - \frac{\hat{\xi}_t^2}{2} \end{bmatrix} \tag{557}$$

$$\underset{n\times 5}{\Sigma_t} = [\sigma_t \quad \hat{\sigma}_t \quad \delta_t \quad \xi_t \quad \hat{\xi}_t]$$

The state vector ω_t can be solved from direct numerical integration of the SDE. However, this becomes infeasible for a large number of simulation paths. Alternatively, we use the fact that the ω_t is Markovian and follows a joint normal distribution. Its mean $M_{(t|v)} = \mathbb{E}_v^T[\omega_t]$ and variance $V_{(t|v)} = \mathbb{V}_v^T[\omega_t]$ at time t conditional on the state at time v for $s < v < t$ are given by the following ODE's

$$\begin{aligned} dM_{(t|v)} &= \left(H_t M_{(t|v)} + h_t - \Sigma_t' b_{t,T}\right) dt \\ dV_{(t|v)} &= \mathbb{V}_v^T[\omega_{t+dt}] - \mathbb{V}_v^T[\omega_t] = \mathbb{V}_v^T[\omega_t + d\omega_t] - \mathbb{V}_v^T[\omega_t] \\ &= \mathbb{V}_v^T[\omega_t] + \mathbb{V}_v^T[d\omega_t, \omega_t] + \mathbb{V}_v^T[\omega_t, d\omega_t] + \mathbb{V}_v^T[d\omega_t] - \mathbb{V}_v^T[\omega_t] \\ &= \left(H_t V_{(t|v)} + V_{(t|v)H_t'} + \Sigma_t' \Sigma_t\right) dt \end{aligned} \tag{558}$$

$$\text{Initial condition:} \quad M_{(v|v)} = \omega_v, \qquad V_{(v|v)} = 0$$

Based on our previous derivation, we have

$$M_{(t|v)} = \mathbb{E}_v^T \begin{bmatrix} z_t \\ \hat{z}_t \\ \log x_t \\ \log q_t \\ \log \hat{q}_t \end{bmatrix} = \begin{bmatrix} E_{v,t} z_v - \int_v^t \beta_{u,t}' b_{u,T} du \\ \hat{E}_{v,t} \hat{z}_v - \int_v^t \hat{\beta}_{u,t}' \left(\delta_u + b_{u,T}\right) du \\ \log \frac{x_v \hat{P}_{v,t}}{P_{v,t}} - \int_v^t \left(\frac{\left(\delta_u - \hat{b}_{u,t}\right)^2 - b_{u,t}^2}{2} + \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' b_{u,T} \right) du \\ \log \frac{q_v}{P_{v,t}} + \int_v^t \left(\frac{b_{u,t}^2 - \xi_u^2}{2} - \eta_u - \left(\xi_u + b_{u,t}\right)' b_{u,T} \right) du \\ \log \frac{\hat{q}_v}{\hat{P}_{v,t}} + \int_v^t \left(\frac{\hat{b}_{u,t}^2 - \hat{\xi}_u^2}{2} - \hat{\eta}_u - \left(\hat{\xi}_u + \hat{b}_{u,t}\right)' \left(\delta_u + b_{u,T}\right) \right) du \end{bmatrix} \tag{559}$$

Since the bond price admits an affine term structure, for example[1], $\log P_{v,t} = -A_{v,t} - B_{v,t} z_v$, we can write the conditional mean in matrix form, such that

$$M_{(t|v)} = I_{v,t}\omega_v + J_{v,t} \qquad \text{where}$$

$$\underset{5\times 5}{I_{v,t}} = \begin{bmatrix} E_{v,t} & 0 & 0 & 0 & 0 \\ 0 & \hat{E}_{v,t} & 0 & 0 & 0 \\ B_{v,t} & -\hat{B}_{v,t} & 1 & 0 & 0 \\ B_{v,t} & 0 & 0 & 1 & 0 \\ 0 & \hat{B}_{v,t} & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{5\times 1}{J_{v,t}} = \begin{bmatrix} -\int_v^t \beta'_{u,t} b_{u,T} du \\ -\int_v^t \hat{\beta}'_{u,t}\left(\delta_u + b_{u,T}\right) du \\ A_{v,t} - \hat{A}_{v,t} - \int_v^t \left(\frac{\left(\delta_u - \hat{b}_{u,t}\right)^2 - b_{u,t}^2}{2} + \left(\delta_u + b_{u,t} - \hat{b}_{u,t}\right)' b_{u,T}\right) du \\ A_{v,t} + \int_v^t \left(\frac{b_{u,t}^2 - \xi_u^2}{2} - \eta_u - \left(\xi_u + b_{u,t}\right)' b_{u,T}\right) du \\ \hat{A}_{v,t} + \int_v^t \left(\frac{\hat{b}_{u,t}^2 - \hat{\xi}_u^2}{2} - \hat{\eta}_u - \left(\hat{\xi}_u + \hat{b}_{u,t}\right)'\left(\delta_u + b_{u,T}\right)\right) du \end{bmatrix} \tag{560}$$

Both $I_{v,t}$ and $J_{v,t}$ are independent of state vector ω_v. They can be pre-calculated and used repeatedly to evolve one step of the state vector for simulation paths. The conditional variance is also independent of the state vector ω_v and can be obtained via numerical integration of the ODE

$$dV_{(t|v)} = \left(H_t V_{(t|v)} + V_{(t|v)H'_t} + \Sigma'_t \Sigma_t\right) dt, \qquad V_{(v|v)} = 0 \tag{561}$$

[1] Note that because the bond price is a *forward looking* of interest rate dynamics, the bond price formula derived under risk neutral measure is still applicable here, even though it depends on the z_v that is evolved under domestic T-forward measure.

11. LIBOR MARKET MODEL

11.1. Introduction

The challenge in modeling interest rates is the existence of a term structure of interest rates embodied in the shape of the forward curve. Fixed income instruments typically depend on a segment of the forward curve rather than a single point. Pricing such instruments requires thus a model describing a stochastic time evolution of the entire forward curve.

There exist a large number of term structure models based on different choices of state variables parameterizing the curve, number of dynamic factors, volatility smile characteristics, etc. The industry standard for interest rates modeling that has emerged since 1997 is the Libor market model [59]. Unlike the older approaches (short rate models), where the underlying state variable is an unobservable instantaneous rate, LMM captures the dynamics of the entire curve of interest rates by using the market observable Libor forwards as its state variables whose volatilities are naturally linked to traded contracts. The time evolution of the forwards is given by a set of intuitive stochastic differential equations in a way which guarantees no-arbitrage of the process. The model is intrinsically multi-factor, meaning that it captures accurately various aspects of the curve dynamics: parallel shift, steepening, butterflies, etc.

On the downside, LMM is far less tractable than, for example, the Hull-White model. In addition, it is in general not Markovian (unless the volatility function is assumed to be separable into time and maturity dependent factors) as opposed to short rate models. As a consequence, all valuations based on LMM have to be done by means of Monte Carlo (MC) simulations.

In this section we will discuss the classic LMM with a local volatility specification. The Libor market model can be regarded as a collection of Black models connected by HJM no-arbitrage condition. It is a discrete version of the HJM model with a lognormal assumption for forward rates.

11.2. Dynamics of the Libor Market Model

The (79) shows that each Libor forward rate $L_{t,i}$ is modeled as a continuous time stochastic process driven by a Brownian motion dW_t^i under the measure $\mathbb{Q}^i$. To make it more general, we assume the rate dynamics is driven by a d-dimensional independent Brownian motion associated with a d-dimensional volatility process

$$dL_{t,i} = L_{t,i}\sigma'_{t,i}dW_t^i \tag{562}$$

The point of switching from scalar-valued to vector-valued Brownian motion is to simplify the handling of correlation structure, which can be implied in the covariance, i.e., the dot product of two volatility vectors. Later we will show that the two formulations are actually equivalent. We know the numeraires for the measure $\mathbb{Q}^{i-1}$ and for the next measure $\mathbb{Q}^i$, so we can explicitly calculate the likelihood process by (25)

$$Z_t := \frac{d\mathbb{Q}^{i-1}}{d\mathbb{Q}^i} = \frac{\frac{P_{t,i-1}}{P_{0,i-1}}}{\frac{P_{t,i}}{P_{0,i}}} = \frac{P_{0,i}}{P_{0,i-1}}\frac{P_{t,i-1}}{P_{t,i}} = \frac{P_{0,i}}{P_{0,i-1}}\left(1+\tau_i L_{t,i}\right) \tag{563}$$

Then its differential form is

$$dZ_t = \frac{P_{0,i}}{P_{0,i-1}}\tau_i dL_{t,i} = \frac{P_{0,i}}{P_{0,i-1}}\tau_i L_{t,i}\sigma'_{t,i}dW_t^i = Z_t\frac{P_{t,i}}{P_{t,i-1}}\tau_i L_{t,i}\sigma'_{t,i}dW_t^i = Z_t\frac{\tau_i L_{t,i}\sigma'_{t,i}}{1+\tau_i L_{t,i}}dW_t^i \tag{564}$$

By examining the integral form of (564), it is easy to write the corresponding θ_t process as in the Girsanov's Theorem

$$\theta_t = -\frac{\tau_i L_{t,i}\sigma_{t,i}}{1+\tau_i L_{t,i}} \tag{565}$$

According to (13), we therefore have the drift adjustment between the two Brownian motions under $\mathbb{Q}^{i-1}$ and $\mathbb{Q}^i$ measure [60]

$$dW_t^i = dW_t^{i-1} + \frac{\tau_i L_{t,i}\sigma_{t,i}}{1+\tau_i L_{t,i}}dt \tag{566}$$

We may seek another way to derive the drift adjustment. Let us first consider two zero coupon bonds maturing at T_{i-1} and T_i respectively. Their price dynamics are given by

$$dP_{t,i-1} = P_{t,i-1}r dt + P_{t,i-1}\delta'_{t,i-1}d\widetilde{W}_t \tag{567}$$

$$dP_{t,i} = P_{t,i} r dt + P_{t,i} \delta'_{t,i} d\widetilde{W}_t$$

We also have

$$P_{t,i-1} = P_{t,i}\left(1 + \tau_i L_{t,i}\right) \tag{568}$$

Differentiating (568) gives

$$dP_{t,i-1} = dP_{t,i} + \tau_i L_{t,i} dP_{t,i} + \tau_i P_{t,i} dL_{t,i} + \tau_i dP_{t,i} dL_{t,i} \tag{569}$$

Since the change of numeraire does not involve the drift terms of the numeraires, we just collect the volatility terms from (569)

$$P_{t,i-1}\delta'_{t,i-1} d\widetilde{W}_t = \left(1 + \tau_i L_{t,i}\right) P_{t,i} \delta'_{t,i} d\widetilde{W}_t + \tau_i P_{t,i} L_{t,i} \sigma'_{t,i} d\widetilde{W}_t \tag{570}$$

The last term is from the fact that $dL_{t,i} = L_{t,i}\sigma'_{t,i} dW_t^i = (\cdot)dt + L_{t,i}\sigma'_{t,i} d\widetilde{W}_t$, since the change of measure does not alter the volatility. Therefore, the volatility terms have the following relationship

$$\begin{aligned} &\left(1 + \tau_i L_{t,i}\right) P_{t,i} \delta_{t,i-1} = \left(1 + \tau_i L_{t,i}\right) P_{t,i} \delta_{t,i} + \tau_i P_{t,i} L_{t,i} \sigma_{t,i} \\ &\Longrightarrow \left(1 + \tau_i L_{t,i}\right)\left(\delta_{t,i-1} - \delta_{t,i}\right) = \tau_i L_{t,i} \sigma_{t,i} \\ &\Longrightarrow \delta_{t,i-1} - \delta_{t,i} = \frac{\tau_i L_{t,i} \sigma_{t,i}}{1 + \tau_i L_{t,i}} \end{aligned} \tag{571}$$

or given that $\delta_{t,t} = 0$ then

$$\delta_{t,i} = -\sum_{j=1}^{i} \frac{\tau_j L_{t,j} \sigma_{t,j}}{1 + \tau_j L_{t,j}} \tag{572}$$

which is equivalent to the bond volatility in (223). Accorder to (30), we have

$$dW_t^i = dW_t^{i-1} + \frac{\tau_i L_{t,i} \sigma_{t,i}}{1 + \tau_i L_{t,i}} dt \tag{573}$$

which is identical to (566). Based on the above analysis, we can recursively write the rate dynamics under T_k-forward measure $\mathbb{Q}^k$ for any given $k \geq 0$

$$dL_{t,i} = \begin{cases} L_{t,i}\sigma'_{t,i}dW_t^k + L_{t,i}\sigma'_{t,i}\displaystyle\sum_{j=k+1}^{i}\frac{\tau_j L_{t,j}\sigma_{t,j}}{1+\tau_j L_{t,j}}dt & \text{if } i > k \text{ and } t \le T_k \\ L_{t,i}\sigma'_{t,i}dW_t^k & \text{if } i = k \text{ and } t \le T_{i-1} \\ L_{t,i}\sigma'_{t,i}dW_t^k - L_{t,i}\sigma'_{t,i}\displaystyle\sum_{j=i+1}^{k}\frac{\tau_j L_{t,j}\sigma_{t,j}}{1+\tau_j L_{t,j}}dt & \text{if } i < k \text{ and } t \le T_{i-1} \end{cases} \tag{574}$$

If we consider a special case for $T_{\eta-1} \le t < T_\eta$ (that is, function η_t is the lowest tenor index such that $t < T_\eta$), the rate dynamics under $\mathbb{Q}^\eta$ is

$$dL_{t,i} = L_{t,i}\sigma'_{t,i}dW_t^\eta + L_{t,i}\sigma'_{t,i}\sum_{j=\eta+1}^{i}\frac{\tau_j L_{t,j}\sigma_{t,j}}{1+\tau_j L_{t,j}}dt, \qquad \forall\, i > \eta \text{ and } t \le T_\eta \tag{575}$$

The rate dynamics in (575) evolves under a sequence of successive forward measures, jumping from $\mathbb{Q}^k$ to $\mathbb{Q}^{k+1}$ when time t advances from current period $[T_{k-1}, T_k)$ into next period $[T_k, T_{k+1})$. This is equivalent to working under a so-called *spot measure*, for which the numeraire is the discretely compounded money market account $\mathcal{M}_t$ consisting of rolled up zeros (an analogue of the money market account M_t whose value inflates continuously at spot rate r_t). This numeraire is defined recursively as follows

$$T_0 \overbrace{\underbrace{\longrightarrow \cdots \longrightarrow T_{\eta-1} \longrightarrow t}_{\mathcal{M}_t} \underbrace{\longrightarrow}_{P_{t,\eta}}}^{\mathcal{M}_\eta} T_\eta$$

$$\mathcal{M}_0 = 1$$

$$\mathcal{M}_i = \mathcal{M}_{i-1}\left(1+\tau_i L_{i-1,i}\right) = \prod_{k=1}^{i}\left(1+\tau_k L_{k-1,k}\right) \quad , \quad 1 \le i \le n \tag{576}$$

$$\mathcal{M}_t = P_{t,\eta}\mathcal{M}_\eta \qquad , \qquad t \in \left[T_{\eta-1}, T_\eta\right)$$

Since the $L_{j-1,j}$, $j = 1, \cdots, \eta$ are all known at t, the randomness of $\mathcal{M}_t$ for $t \in \left[T_{\eta-1}, T_\eta\right)$ is solely determined by the zero price $P_{t,\eta}$, which is also the numeraire of the successive forward measure $\mathbb{Q}^\eta$. The common source of the stochasticity of the numeraires suggests that the two associated measures are actually equivalent. In fact, if we take the limit of $\mathcal{M}_t$ by varying the size of the time intervals τ_i, for example, if we extend the τ_η such that $T_\eta = T$, the spot measure becomes the T-forward measure. On the other hand, if the time intervals τ_i approach to zero, the $\mathcal{M}_t$ becomes a continuously compounded money market account M_t, and therefore the spot measure coincides with the usual risk-neutral measure. In the latter case, the forward Libor rate $L_{t,i}$ degenerates into an instantaneous forward rate $f_{t,T}$ (to ease notation we denote T_i by T) and its instantaneous volatility becomes $f_{t,T}\sigma_{t,T}$. By writing W_t^η to $\widetilde{W}_t$, we can rewrite (575) as

$$df_{t,T} = f_{t,T}\sigma_{t,T}'\left(d\widetilde{W}_t + \left(\int_t^T \frac{f_{t,u}\sigma_{t,u}du}{1 + f_{t,u}du}\right)dt\right) = f_{t,T}\sigma_{t,T}'\left(d\widetilde{W}_t + \left(\int_t^T f_{t,u}\sigma_{t,u}du\right)dt\right) \tag{577}$$

Let us define $\beta_{t,T} = f_{t,T}\sigma_{t,T}$, then

$$df_{t,T} = \beta_{t,T}'\left(d\widetilde{W}_t + \left(\int_t^T \beta_{t,u}'du\right)dt\right) = \beta_{t,T}'\left(\int_t^T \beta_{t,u}du\right)dt + \beta_{t,T}'d\widetilde{W}_t \tag{578}$$

This is identical to the second formula in (223). In other words, the Libor market model can be regarded as a collection of Black models connected by HJM no-arbitrage condition. It is a discrete version of the HJM model with a lognormal assumption for forward rates.

For simplicity, we revert to the rate dynamics definition as in (79) where each rate is driven by a single Brownian motion. The instantaneous correlation between infinitesimal changes in rates is given by the correlation between the two Brownian motions

$$dW_t^i dW_t^j = \rho_{ij}dt \tag{579}$$

This is indeed consistent with the multi-dimensional rate dynamics definition where we assume the stochastic driver is a d-dimensional independent Brownian motion, in which the instantaneous correlation has been implied in the rate volatilities, that is

$$\rho_{ij} = \frac{\mathrm{Cov}\left(dL_{t,i}, dL_{t,j}\right)}{\mathrm{Std}\left(dL_{t,i}\right)\mathrm{Std}\left(dL_{t,j}\right)} = \frac{\sigma_{t,i}'\sigma_{t,j}}{\sqrt{\sigma_{t,i}'\sigma_{t,i}}\sqrt{\sigma_{t,j}'\sigma_{t,j}}} = \frac{\sigma_{t,i}'\sigma_{t,j}}{\|\sigma_{t,i}\|\|\sigma_{t,j}\|} \tag{580}$$

In the case of 1D Brownian motion driver, the rate dynamics under $\mathbb{Q}^k$ in (574) can be translated into

$$dL_{t,i}=\begin{cases} L_{t,i}\sigma_{t,i}dW_t^k+L_{t,i}\sigma_{t,i}\displaystyle\sum_{j=k+1}^{i}\frac{\tau_jL_{t,j}\rho_{ij}\sigma_{t,j}}{1+\tau_jL_{t,j}}dt & \text{if } i>k \text{ and } t\le T_k \\ L_{t,i}\sigma_{t,i}dW_t^k & \text{if } i=k \text{ and } t\le T_{i-1} \\ L_{t,i}\sigma_{t,i}dW_t^k-L_{t,i}\sigma_{t,i}\displaystyle\sum_{j=i+1}^{k}\frac{\tau_jL_{t,j}\rho_{ij}\sigma_{t,j}}{1+\tau_jL_{t,j}}dt & \text{if } i<k \text{ and } t\le T_{i-1} \end{cases} \tag{581}$$

11.3. Theoretical Incompatibility between LMM and SMM

In theory, the LMM and the SMM are not consistent. That is, if forward Libor rates are lognormal under $\mathbb{Q}^i$ measures, as assumed by LMM, then forward swap rates cannot be lognormal at the same time under $\mathbb{Q}^{a,b}$ measure, as assumed by SMM. This can be illustrated through the following procedure: firstly assume the hypothesis of the LMM, i.e., the forward Libor rates $L_{t,i}$ are lognormal under $\mathbb{Q}^i$ measures, and then apply the change of measure to obtain their dynamics under the $\mathbb{Q}^{a,b}$ measure, then apply Ito lemma to derive the dynamics of swap rate $S_t^{a,b}$ under $\mathbb{Q}^{a,b}$ measure. The derived swap rate dynamics is in fact not lognormal, which is inconsistent with the hypothesis of the SMM. In fact, the (599) (derived later) shows that swap rate can be represented as a sum of weighted forward rates, i.e., $S_t^{a,b}=\sum_{i=a+1}^{b}\omega_{t,i}L_{t,i}$. Since the weights $\omega_{t,i}$ vary much less than the rates, we can roughly treat them as constant. Therefore the sum of lognormal random variables asymptotically resembles a normal distribution by central limit theorem, and cannot still be lognormal.

However, from a practical point of view, this incompatibility seems to weaken. Indeed, Monte Carlo simulation shows that most of times the distribution of $S_t^{a,b}$ is not far from being lognormal. In normal market conditions, the two distributions are hardly distinguishable.

11.4. Instantaneous Correlation and Terminal Correlation

The instantaneous correlation in (580) is a quantity summarizing the degree of dependence between instantaneous changes of different forward Libor rates. Because it is determined only by the diffusion terms, it does not depend on the particular probability measure (or numeraire asset) that is associated. However the terminal correlation, which describes the dependence between the rates rather than their infinitesimal changes, depends on the particular probability measure.

Suppose we have the rate dynamics under $\mathbb{Q}^k$ by (574)

$$dL_{t,i} = L_{t,i}\mu_t^{i,k}dt + L_{t,i}\sigma'_{t,i}dW_t^k$$

$$\mu_t^{i,k} = \begin{cases} \sigma'_{t,i}\sum_{j=k+1}^{i}\frac{\tau_j L_{t,j}\sigma_{t,j}}{1+\tau_j L_{t,j}} & \text{if } i > k \text{ and } t \le T_k \\ 0 & \text{if } i = k \text{ and } t \le T_{i-1} \\ -\sigma'_{t,i}\sum_{j=i+1}^{k}\frac{\tau_j L_{t,j}\sigma_{t,j}}{1+\tau_j L_{t,j}} & \text{if } i < k \text{ and } t \le T_{i-1} \end{cases} \tag{582}$$

and thus the rate at a future time T can be integrated from time t to give

$$L_{T,i} = L_{t,i}\exp\left(\int_t^T \mu_u^{i,k}du - \frac{1}{2}\int_t^T \sigma'_{u,i}\sigma_{u,i}du + \int_t^T \sigma'_{u,i}dW_u^k\right) \tag{583}$$

Since there are rates $L_{t,j}$ in the drift term $\mu_t^{i,k}$ and they are random and path-dependent ($\sigma_{t,j}$ is deterministic though), the $\mu_t^{i,k}$ must also be random and non-markovian. This complicates the calculation of the rate expectation under different measures. However because the randomness of $\mu_t^{i,k}$ is negligible compared to the diffusion term in the rate dynamics, we can approximate it by freezing its rates dependence to be at t for $t < u$

$$\bar{\mu}_u^{i,k} = \begin{cases} \sigma'_{u,i}\sum_{j=k+1}^{i}\frac{\tau_j L_{t,j}\sigma_{u,j}}{1+\tau_j L_{0,j}} & \text{if } i > k \text{ and } u \le T_k \\ 0 & \text{if } i = k \text{ and } u \le T_{i-1} \\ -\sigma'_{u,i}\sum_{j=i+1}^{k}\frac{\tau_j L_{t,j}\sigma_{u,j}}{1+\tau_j L_{t,j}} & \text{if } i < k \text{ and } u \le T_{i-1} \end{cases} \tag{584}$$

Under this approximation the rate is

$$L_{T,i} \approx L_{t,i}\exp\left(\int_t^T \bar{\mu}_u^{i,k}du - \frac{1}{2}\int_t^T \sigma'_{u,i}\sigma_{u,i}du + \int_t^T \sigma'_{u,i}dW_u^k\right) \tag{585}$$

then one can easily write its expectation

$$\mathbb{E}_t^k\left[L_{T,i}\right] \approx L_{t,i} \exp\left(\int_t^T \bar{\mu}_u^{i,k} du\right) \tag{586}$$

and moreover

$$\begin{aligned}
&\frac{\mathbb{E}_t^k\left[L_{T,i}L_{T,j}\right]}{\mathbb{E}_t^k\left[L_{T,i}\right]\mathbb{E}_t^k\left[L_{T,j}\right]} \\
&\approx \frac{\mathbb{E}_t^k\left[L_{t,i}e^{\int_t^T \bar{\mu}_u^{i,k}du-\frac{1}{2}\int_t^T \sigma_{u,i}'\sigma_{u,i}du+\int_t^T \sigma_{u,i}'dW_u^k}L_{t,j}e^{\int_t^T \bar{\mu}_u^{j,k}du-\frac{1}{2}\int_t^T \sigma_{u,j}'\sigma_{u,j}du+\int_t^T \sigma_{u,j}'dW_u^k}\right]}{L_{t,i}e^{\int_t^T \bar{\mu}_u^{i,k}du}L_{t,j}e^{\int_t^T \bar{\mu}_u^{j,k}du}} \\
&= \exp\left(-\frac{1}{2}\int_t^T \sigma_{u,i}'\sigma_{u,i}du - \frac{1}{2}\int_t^T \sigma_{u,j}'\sigma_{u,j}du\right)\mathbb{E}_t^k\left[\exp\left(\int_t^T \left(\sigma_{u,i}+\sigma_{u,j}\right)' dW_u^k\right)\right] \\
&= \exp\left(-\frac{1}{2}\int_t^T \sigma_{u,i}'\sigma_{u,i}du - \frac{1}{2}\int_t^T \sigma_{u,j}'\sigma_{u,j}du + \frac{1}{2}\int_t^T \left(\sigma_{u,i}+\sigma_{u,j}\right)'\left(\sigma_{u,i}+\sigma_{u,j}\right)du\right) \\
&= \exp\left(\int_t^T \sigma_{u,i}'\sigma_{u,j}du\right)
\end{aligned} \tag{587}$$

Hence from the definition of correlation, we can compute the terminal correlation ϱ_{ij} for the period from t to T

$$\begin{aligned}
\varrho_{ij} &= \frac{\mathbb{E}_t^k\left[\left(L_{T,i}-\mathbb{E}_t^k\left[L_{T,i}\right]\right)\left(L_{T,j}-\mathbb{E}_t^k\left[L_{T,j}\right]\right)\right]}{\sqrt{\mathbb{E}_t^k\left[\left(L_{T,i}-\mathbb{E}_t^k\left[L_{T,i}\right]\right)^2\right]}\sqrt{\mathbb{E}_t^k\left[\left(L_{T,j}-\mathbb{E}_t^k\left[L_{T,j}\right]\right)^2\right]}} \\
&= \frac{\mathbb{E}_t^k\left[L_{T,i}L_{T,j}\right]-\mathbb{E}_t^k\left[L_{T,i}\right]\mathbb{E}_t^k\left[L_{T,j}\right]}{\sqrt{\mathbb{E}_t^k\left[L_{T,i}^2\right]-\mathbb{E}_t^k\left[L_{T,i}\right]^2}\sqrt{\mathbb{E}_t^k\left[L_{T,j}^2\right]-\mathbb{E}_t^k\left[L_{T,j}\right]^2}} \\
&\approx \frac{\exp\left(\int_t^T \sigma_{u,i}'\sigma_{u,j}du\right)-1}{\sqrt{\exp\left(\int_t^T \sigma_{u,i}'\sigma_{u,i}du\right)-1}\sqrt{\exp\left(\int_t^T \sigma_{u,j}'\sigma_{u,j}du\right)-1}}
\end{aligned} \tag{588}$$

Considering the covariance integral $\int_t^T \sigma_{u,i}'\sigma_{u,j}du$ is in general much smaller than 1, the above equation can be further simplified to

$$\varrho_{ij} \approx \frac{\int_t^T \sigma'_{u,i}\sigma_{u,j}du}{\sqrt{\int_t^T \sigma'_{u,i}\sigma_{u,i}du}\sqrt{\int_t^T \sigma'_{u,j}\sigma_{u,j}du}} \tag{589}$$

or written in terms of the constant instantaneous correlation with scalar volatilities

$$\varrho_{ij} \approx \frac{\rho_{ij}\int_t^T \sigma_{u,i}\sigma_{u,j}du}{\sqrt{\int_t^T \sigma_{u,i}^2 du}\sqrt{\int_t^T \sigma_{u,j}^2 du}} \tag{590}$$

This formula (a.k.a Rebonato's terminal correlation formula [61]) shows through Schwarz's Inequality that $|\varrho_{ij}| \le |\rho_{ij}|$, the terminal correlations are, in absolute value, always smaller than or equal to instantaneous correlations.

11.5. Parametric Volatility and Correlation Structure

11.5.1. Parametric Instantaneous Volatility

In previous section we have introduced a procedure to bootstrap the caplet implied volatilities from a term structure of cap implied volatility. The relationship between the caplet volatility σ_i and the instantaneous volatility $\sigma_{u,i}$ for the forward Libor rate covering period $T_{i-1}\sim T_i$ is given by (82)

$$\varsigma_i^2(T_{i-1}-t) = \int_t^{T_{i-1}} \sigma_{u,i}^2 du \tag{591}$$

Roughly speaking, the caplet volatility is just an average of the instantaneous volatility over time. A commonly used parametric form of instantaneous volatility is given as [62]

$$\begin{aligned} &\sigma_{t,i} := \phi_i \psi_{t,i}, \qquad t < T_{i-1} \\ &\psi_{t,i} = \psi(T_{i-1}-t; a, b, c, d) = \big(a + b(T_{i-1}-t)\big)e^{-c(T_{i-1}-t)} + d \\ &a + d > 0, \qquad b, c, d > 0 \end{aligned} \tag{592}$$

This linear exponential formulation can be seen to have a parametric core ψ that is locally altered for each maturity T_i by the ϕ_i. The parametric core ψ allows a humped shape occurring in the short end of the volatility curve. The local modifications, if small (i.e., ϕ_i is close to 1), do not destroy the essential dependence on the time to maturity. The formulation also implies the volatilities are close to $(a + d)$ for short term maturities and close to d for the long term maturities. Notice that the extra ϕ_i terms make this form over-parameterized for calibration to caplet volatilities. However, it adds a flexibility that can improve the joint calibration of the model to both the caps and swaptions markets.

In this formulation, we can easily compute the accumulated variance/covariance by a closed form formula with the help of the following indefinite integral [63]

$$
\begin{aligned}
I_t^{i,j} &= \int \psi_{t,j}\psi_{t,j}dt = \int \Big((a-b\delta_i)e^{c\delta_i}+d\Big)\Big((a-b\delta_j)e^{c\delta_j}+d\Big)dt \\
&= \frac{ad}{c}\left(e^{c\delta_i}+e^{c\delta_j}\right)+d^2t-\frac{bd}{c^2}\Big(e^{c\delta_i}(c\delta_i-1)+e^{c\delta_j}(c\delta_j-1)\Big) \\
&+\frac{e^{c(\delta_i+\delta_j)}}{4c^3}\Big(2a^2c^2+2abc(1-c\delta_i-c\delta_j)+b^2(1-c\delta_i-c\delta_j+2c^2\delta_i\delta_j)\Big)
\end{aligned}
\tag{593}
$$

where $\delta_i = t - T_{i-1}$. Then the accumulated covariance is

$$
\int_t^T \rho_{ij}\sigma_{u,i}\sigma_{u,j}du = \rho_{ij}\phi_i\phi_j\left(I_T^{i,j}-I_t^{i,j}\right) \tag{594}
$$

and the i-th caplet volatility is simply

$$
\varsigma_i^2(T_{i-1}-t) = \int_t^T \sigma_{u,i}^2 du = \phi_i^2\left(I_T^{i,i}-I_t^{i,i}\right) \tag{595}
$$

11.5.2. Parametric Instantaneous Correlation

The instantaneous correlation is assumed to be time homogeneous. Other than being symmetric and semi-positive definite, the instantaneous correlation matrix associated with a LMM are expected to have some additional qualities. The main properties are [64]

1. Correlations are typically positive, $\rho_{ij} > 0, \quad \forall\, i,j$.
2. When moving away from diagonal entry $\rho_{ii} = 1$ along a row or a column, correlation should monotonically decrease. Joint movements of faraway rates are less correlated than movements of rates with close maturities.
3. Moving along the zero curve, the larger the tenor, the more correlated moves there will be in adjacent forward rates.

Schoenmakers and Coffey suggest a full rank 3-factor parametric form for the correlation structure [65]

$$\begin{aligned}\rho_{ij} &= \rho(i,j;\beta_1,\beta_2,\beta_3,m)\\ &= \exp\left(-\frac{|i-j|}{m-1}\left(-\log\beta_1+\beta_2\frac{i^2+j^2+ij-3(m-1)(i+j)+2m^2-m-4}{(m-2)(m-3)}\right.\right.\\ &\qquad\left.\left.-\beta_3\frac{i^2+j^2+ij-(m+3)(i+j)+3m^2+2}{(m-2)(m-3)}\right)\right)\end{aligned} \tag{596}$$

$$0<\beta_3<3\beta_2,\qquad 0<\beta_2+\beta_3<-\log\beta_1$$

where m denotes the total number of forward Libor rates under consideration. The calibration experiments of Schoenmakers and Coffey show that the above correlation structure suits very well in practice. However, calibrating a 3-parameter matrix takes longer time than a 2-parameter one. Furthermore, the experiments of implied calibration reveal that the calibrated β_3 is practically always close to 0. Thus they suggest to set $\beta_3=0$ and adopt the following computationally improved correlation structure

$$\begin{aligned}\rho_{ij} &= \rho(i,j;\beta_1,\beta_2,m)\\ &= \exp\left(-\frac{|i-j|}{m-1}\left(-\log\beta_1+\beta_2\frac{i^2+j^2+ij-3(m-1)(i+j)+2m^2-m-4}{(m-2)(m-3)}\right)\right)\end{aligned} \tag{597}$$

$$0<\beta_2<-\log\beta_1$$

11.6. Analytical Approximation of Swaption Volatilities

At time t, the Black swaption volatility (squared and multiplied by T_a-t) is given by

$$v_t^{a,b}=\left(\varsigma_t^{a,b}\right)^2(T_a-t)=\int_t^{T_a}\left(\sigma_u^{a,b}\right)^2du=\int_t^{T_a}\left(d\log S_u^{a,b}\right)^2du \tag{598}$$

We can derive a formula to compute the volatility under a few approximations. Firstly, we write the swap rate in (77) as a linear combination of forward rates

$$S_t^{a,b}=\sum_{i=a+1}^{b}\omega_{t,i}L_{t,i} \tag{599}$$

where the weights

$$\omega_{t,i}=\frac{\tau_i P_{t,i}}{\sum_{k=a+1}^{b}\tau_k P_{t,k}}=\frac{\tau_i\prod_{j=a+1}^{i}\frac{1}{1+\tau_j L_{t,j}}}{\sum_{k=a+1}^{b}\tau_k\prod_{j=a+1}^{k}\frac{1}{1+\tau_j L_{t,j}}} \tag{600}$$

In fact the existence of (599) implies the aforementioned incompatibility between LMM and SMM under a single measure, because a sum of lognormal distributions cannot be lognormal. However, as noted previously, this approximation is not bad at all. By differentiating both sides of (599), we have

$$dS_t^{a,b}=(\cdot)dt+\sum_{i=a+1}^{b}\left(\omega_{t,i}dL_{t,i}+L_{t,i}d\omega_{t,i}\right) \tag{601}$$

and then

$$d\log S_t^{a,b}=\frac{dS_t^{a,b}}{S_t^{a,b}}-\frac{\left(dS_t^{a,b}\right)^2}{2\left(S_t^{a,b}\right)^2}=(\cdot)dt+\frac{1}{S_t^{a,b}}\sum_{i=a+1}^{b}\left(\omega_{t,i}dL_{t,i}+L_{t,i}d\omega_{t,i}\right) \tag{602}$$

The above equation can be squared to give

$$\begin{aligned}\left(d\log S_t^{a,b}\right)^2&=\frac{1}{\left(S_t^{a,b}\right)^2}\left(\sum_{i=a+1}^{b}\left(\omega_{t,i}dL_{t,i}+L_{t,i}d\omega_{t,i}\right)\right)^2\\&=\frac{1}{\left(S_t^{a,b}\right)^2}\sum_{i,j=a+1}^{b}\big(\omega_{t,i}\omega_{t,j}dL_{t,i}dL_{t,j}+L_{t,i}L_{t,j}d\omega_{t,i}d\omega_{t,j}+\omega_{t,i}L_{t,j}dL_{t,i}d\omega_{t,j}\\&\qquad+\omega_{t,j}L_{t,i}dL_{t,j}d\omega_{t,i}\big)\end{aligned} \tag{603}$$

Considering that the variability of the $\omega_{t,i}$ is much smaller than that of the $L_{t,i}$, the above equation can be simplified to

$$\begin{aligned}\left(d\log S_t^{a,b}\right)^2&\approx\frac{1}{\left(S_t^{a,b}\right)^2}\sum_{i,j=a+1}^{b}\omega_{t,i}\omega_{t,j}dL_{t,i}dL_{t,j}\\&=\frac{1}{\left(S_t^{a,b}\right)^2}\sum_{i,j=a+1}^{b}\omega_{t,i}\omega_{t,j}L_{t,i}L_{t,j}\sigma_{t,i}\sigma_{t,j}dW_t^i dW_t^j\end{aligned} \tag{604}$$

$$= \frac{1}{\left(S_t^{a,b}\right)^2} \sum_{i,j=a+1}^{b} \omega_{t,i}\omega_{t,j}L_{t,i}L_{t,j}\rho_{ij}\sigma_{t,i}\sigma_{t,j}dt$$

Hence the Black swaption volatility is given by

$$v_t^{a,b} = \int_t^{T_a} \left(d\log S_u^{a,b}\right)^2 du \approx \int_t^{T_a} \left(\frac{1}{\left(S_u^{a,b}\right)^2} \sum_{i,j=a+1}^{b} \omega_{u,i}\omega_{u,j}L_{u,i}L_{u,j}\rho_{ij}\sigma_{u,i}\sigma_{u,j} \right) du \tag{605}$$

The above quantity is path-dependent and can only be evaluated by Monte Carlo simulations. In order to gain an analytical estimation, we make a further approximation by freezing all $L_{u,i}$ to their values at t and hence we reach the Rebonato formula [66]

$$v_t^{a,b} \approx \frac{1}{\left(S_t^{a,b}\right)^2} \sum_{i,j=a+1}^{b} \omega_{t,i}\omega_{t,j}L_{t,i}L_{t,j}\rho_{ij} \int_t^{T_a} \sigma_{u,i}\sigma_{u,j}du \tag{606}$$

Hull and White have proposed another approximation formula which is slightly more sophisticated than the above one. Because $\omega_{t,i}$ is defined as a function of forward rates $L_{t,j}$, its total differential can be expressed as a sum of partial differentials

$$d\omega_{t,i} = \sum_{j=a+1}^{b} \frac{\partial \omega_{t,i}}{\partial L_{t,j}} dL_{t,j} \tag{607}$$

Then we can estimate

$$\begin{aligned} dS_t^{a,b} &= (\cdot)dt + \sum_{i=a+1}^{b} \left(\omega_{t,i}dL_{t,i} + L_{t,i}d\omega_{t,i}\right) \\ &= (\cdot)dt + \sum_{i=a+1}^{b} \omega_{t,i}dL_{t,i} + \sum_{i,j=a+1}^{b} L_{t,i}\frac{\partial \omega_{t,i}}{\partial L_{t,j}}dL_{t,j} \\ &= (\cdot)dt + \sum_{j=a+1}^{b} \left(\omega_{t,j} + \sum_{i=a+1}^{b} L_{t,i}\frac{\partial \omega_{t,i}}{\partial L_{t,j}}\right)dL_{t,j} \\ &= (\cdot)dt + \sum_{j=a+1}^{b} \bar{\omega}_{t,j}dL_{t,j} \end{aligned} \tag{608}$$

in terms of a set of new weights

$$\bar{\omega}_{t,j} = \omega_{t,j} + \sum_{i=a+1}^{b} L_{t,i} \frac{\partial \omega_{t,i}}{\partial L_{t,j}} \tag{609}$$

The way to estimate the partial derivatives is as follows. We know that

$$\frac{\partial P_{t,i}}{\partial L_{t,j}} = -\frac{P_{t,i}\tau_j}{1+\tau_j L_{t,j}} \, 1_{\{i\geq j\}} \tag{610}$$

where $1_{\{i\geq j\}}$ is an indicator function that equals to 1 if $i \geq j$ and 0 otherwise. Therefore, the partial derivative

$$\begin{aligned}
\frac{\partial \omega_{t,i}}{\partial L_{t,j}} &= \frac{\partial \dfrac{\tau_i P_{t,i}}{\sum_{k=a+1}^{b} \tau_k P_{t,k}}}{\partial L_{t,j}} \\
&= -\frac{\tau_i P_{t,i}}{\sum_{k=a+1}^{b} \tau_k P_{t,k}} \frac{\tau_j}{1+\tau_j L_{t,j}} \, 1_{\{i\geq j\}} - \frac{\tau_i P_{t,i}}{\left(\sum_{k=a+1}^{b} \tau_k P_{t,k}\right)^2} \frac{\partial \sum_{k=a+1}^{b} \tau_k P_{t,k}}{\partial L_{t,j}} \\
&= -\frac{\omega_{t,i}\tau_j}{1+\tau_j L_{t,j}} \, 1_{\{i\geq j\}} + \frac{\omega_{t,i}}{\sum_{k=a+1}^{b} \tau_k P_{t,k}} \frac{\tau_j \sum_{k=j}^{b} \tau_k P_{t,k}}{1+\tau_j L_{t,j}} \\
&= \frac{\omega_{t,i}\tau_j}{1+\tau_j L_{t,j}} \left(\frac{A_t^{j-1,b}}{A_t^{a,b}} - 1_{\{i\geq j\}} \right)
\end{aligned} \tag{611}$$

Hence the Black swaption volatility is given by

$$\bar{v}_t^{a,b} = \int_t^{T_a} \left(d \log S_u^{a,b}\right)^2 du = \int_t^{T_a} \left(\frac{1}{\left(S_u^{a,b}\right)^2} \sum_{i,j=a+1}^{b} \bar{\omega}_{u,i} \bar{\omega}_{u,j} L_{u,i} L_{u,j} \rho_{ij} \sigma_{u,i} \sigma_{u,j} \right) du \tag{612}$$

Again by freezing all $L_{u,i}$ to their values at t, we reach the Hull-White formula [67]

$$\bar{v}_t^{a,b} \approx \frac{1}{\left(S_t^{a,b}\right)^2} \sum_{i,j=a+1}^{b} \bar{\omega}_{t,i} \bar{\omega}_{t,j} L_{t,i} L_{t,j} \rho_{ij} \int_t^{T_a} \sigma_{u,i} \sigma_{u,j} du \tag{613}$$

In most situations the two approximations differ negligibly. They work equally well and both give satisfactory estimation in general.

11.7. Calibration of LMM

Calibrating the LMM means reducing the distance between the market quotes (e.g., for caps/floors and swaptions) and the prices obtained in the model by working on the model parameters. Though the zero curve is also a market input, it will be automatically fitted and implied in the price calculation. In the LMM framework, the free parameters are those deriving from the instantaneous correlation and volatility parameterizations. In our example, the instantaneous volatility is assumed to be defined by (592)

$$\begin{aligned}&\sigma_{t,i} = \phi_i \psi_{t,i} = \phi_i\big([a + b(T_{i-1} - t)]e^{-c(T_{i-1}-t)} + d\big), \qquad t < T_{i-1}, i = 2, \cdots, m \\ &a + d > 0, \qquad b, c, d > 0, \qquad 0.9 < \phi_i < 1.1\end{aligned} \tag{614}$$

and the instantaneous correlation is assumed to be given by (597)

$$\begin{aligned}\rho_{ij} &= \rho(i, j; \beta_1, \beta_2, m) \\ &= \exp\left(-\frac{|i-j|}{m-1}\left(-\log\beta_1 + \beta_2 \frac{i^2 + j^2 + ij - 3(m-1)(i+j) + 2m^2 - m - 4}{(m-2)(m-3)}\right)\right) \\ &0 < \beta_2 < -\log\beta_1\end{aligned} \tag{615}$$

where m is the total number of forward rates under consideration. To ease the notation, we define the volatility parameter vectors with their constraints as $\alpha = (a, b, c, d)' \in \mathbb{C}_\alpha$ and $\phi = (\phi_2, \cdots, \phi_m)' \in \mathbb{C}_\phi$, and the correlation parameter vector as $\beta = (\beta_1, \beta_2)' \in \mathbb{C}_\beta$.

In the classical LMM, the volatility skews and smiles are not considered, hence the calibration only applies to the ATM volatilities in our example. However, the model has evolved remarkably to relax such limitation along with other issues in recent years [68] [69].

11.7.1. Instantaneous Correlation: Inputs or Outputs

Per previous discussion, the swaption market quotes have implied correlation information. However, should we infer the correlation structure *endogenously* from the swaption market quotes or should we estimate *exogenously* and impose it into the model leaving the calibration only to volatility parameters? The answer surely depends on the quality of the market data as well as the application of the model. As Jackel and Rabonato pointed [63], European swaption prices turn out to be relatively insensitive to instantaneous (rather than terminal) correlation details, which means the correlation parameters implied from swaption quotes may be unstable and erratic. It would be wise to impose a good exogenous instantaneous correlation structure and subsequently play on volatilities to calibrate. This allows us not only to incorporate the behavior of the real market rates in the model but also to unburden the optimization procedure.

Instantaneous correlation matrix can be estimated using historical market data of rates. Brigo has done some tests on the stability of the estimates, showing that the values remain rather constant even if the sample size or the time window is changed [70]. The historically estimated matrix can then be fitted or smoothed by a chosen parametric correlation function by minimizing some loss function of the difference between the two matrices. Such a more regular correlation structure can lead, through calibration, to more regular volatilities and to a more stable evolution of the volatility term structure.

For demonstration purpose, we will make the calibration more general by taking the correlation as a model calibration output instead.

11.7.2. Joint Calibration to Caplets and Swaptions by Global Optimization

The model can be calibrated jointly to a term structure of ATM caplet volatility and a matrix of ATM swaption volatilities. Generally traders translate swaption prices into implied Black's swaption volatilities and organize them in a table where the rows are indexed by the option maturity time and the columns are indexed by the length of the underlying swap. The calibration is equivalent to the following optimization process performed over α, ϕ and β

$$\underset{\alpha,\phi,\beta}{\operatorname{Argmin}}\left(w_{cpl}\sum_{2\le i\le b} w_i^{cpl}\left(\varsigma_i^{mkt}-\varsigma_i^{mdl}\right)^2+w_{swpt}\sum_{1\le i<j\le b} w_{i,j}^{swpt}\left(\varsigma_{i,j}^{mkt}-\varsigma_{i,j}^{mdl}\right)^2\right) \tag{616}$$

subject to $\alpha\in\mathbb{C}_\alpha,\;\; \phi\in\mathbb{C}_\phi,\;\; \beta\in\mathbb{C}_\beta$

where w_{cpl} and w_{swpt} are the weights to the cap and swaption markets respectively, w_i^{cpl} and $w_{a,b}^{swpt}$ are the weights to each caplet and swaption and the summations are made over the set of considered caplets and swaptions.

The calibration procedure follows two steps. Firstly we calibrate the time-homogeneous part $\psi_{t,i}$ of the volatility function along with the correlation function

$$\underset{\alpha,\beta}{\operatorname{Argmin}}\left(w_{cpl}\sum_{2\le i\le b} w_i^{cpl}\left(\varsigma_i^{mkt}-\varsigma_i^{mdl}\right)^2+w_{swpt}\sum_{1\le i<j\le b} w_{i,j}^{swpt}\left(\varsigma_{i,j}^{mkt}-\varsigma_{i,j}^{mdl}\right)^2\right) \tag{617}$$

subject to $\alpha\in\mathbb{C}_\alpha,\ \ \phi=1,\ \ \beta\in\mathbb{C}_\beta$

The above minimization implies a suitable fit for the time-homogenous volatility function, e.g., $\alpha=\hat{\alpha}$, given a set of weights. Unfortunately there is, in a general case, not enough degrees of freedom left for perfect fit of all the considered caplets and swaptions. Another constrained optimization problem with the vector ϕ as variables is therefore solved

$$\underset{\phi,\beta}{\operatorname{Argmin}}\left(w_{cpl}\sum_{2\le i\le b} w_i^{cpl}\left(\varsigma_i^{mkt}-\varsigma_i^{mdl}\right)^2+w_{swpt}\sum_{1\le i<j\le b} w_{i,j}^{swpt}\left(\varsigma_{i,j}^{mkt}-\varsigma_{i,j}^{mdl}\right)^2\right) \tag{618}$$

subject to $\alpha=\hat{\alpha},\ \ \phi\in\mathbb{C}_\phi,\ \ \beta\in\mathbb{C}_\beta$

where the weights, if wanted, might be changed from the previous optimization. Preferably one chooses quite general weights in the first optimization and then tries to fit valid caplets and swaptions as good as possible with the help of the ϕ.

11.7.3. Calibration to Co-terminal Swaptions

A Bermudan swaption contract denoted by "y-non-call-x" gives the holder the right to enter into a swap with a prescribed strike rate K at any time T_i , $i=a,\cdots,b-1$ where $T_a=x$ and $T_b=y$. The first exercise opportunity in this case would be at T_a or x years after inception. The swap that can be entered into has always the same terminal maturity, namely T_b or y years after inception, independent on when exercise takes place. A Bermudan swaption that entitles the holder to enter into a swap that pays the fixed rate is known as payer's, otherwise as receiver's. Since Bermudan swaptions are useful hedges for callable bonds, they are actively traded and one of the most liquid fixed income derivatives with built-in early exercise features.

Bermudan swaptions are typically hedged with the corresponding co-terminal ATM European swaptions. Co-terminal means that the swaptions though may have different option maturities, their underlying swaps mature at the same time, e.g., T_b. For this reason, a calibration procedure has been introduced [71], which calibrates the model exactly to the co-terminal swaptions, while achieving a satisfactory fit to the upper triangular portion of the volatility matrix. The calibration is based on a recursive algorithm, which will be described as follows.

The market volatility $v_t^{k,b}$ (squared and multiplied by $T_k - t$) of co-terminal swaption maturing at T_k can be approximated analytically based on (606) or (613)

$$\left(v_t^{k,b} S_t^{k,b}\right)^2 = \sum_{i=k+1}^{b} \sum_{j=k+1}^{b} F_{t,i} F_{t,j} \phi_i \phi_j \delta_{t,k}^{i,j}, \qquad \forall\, k = 1, \cdots, y-1$$
$$\text{where } F_{t,i} = \omega_{t,i}^{k,b} L_{t,i} \text{ and } \delta_{t,k}^{i,j} = \rho_{ij} \int_t^{T_k} \psi_{u,i} \psi_{u,j} du \tag{619}$$

We can transform (619) to a quadratic form for variable ϕ_{k+1} by rearranging the terms in the double summation

$$\underbrace{F_{t,k+1}^2 \delta_{t,k}^{k+1,k+1}}_{A} \boldsymbol{\phi_{k+1}^2} + \underbrace{2F_{t,k+1} \sum_{i=k+2}^{b} F_{t,i} \phi_i \delta_{t,k}^{i,k+1}}_{B} \boldsymbol{\phi_{k+1}}$$
$$+ \underbrace{\sum_{i=k+2}^{b} \sum_{j=k+2}^{b} F_{t,i} F_{t,j} \phi_i \phi_j \delta_{t,k}^{i,j} - \left(v_t^{k,b} S_t^{k,b}\right)^2}_{C} = 0 \tag{620}$$

We first consider the co-terminal swaption maturing at T_{b-1} for $k = b - 1$, the same maturity as the Bermudan swaption. This is the last co-terminal swaption, and its underlying is a one-period swap, $S_t^{b-1,b}$. Given an initial guess of α and β, the $\delta_{t,b-1}^{b,b}$ is determined, we can write (620) as

$$F_{t,b}^2 \delta_{t,b-1}^{b,b} \boldsymbol{\phi_b^2} - \left(v_t^{b-1,b} S_t^{b-1,b}\right)^2 = 0 \tag{621}$$

Since $\omega_{t,b}^{b-1,b} = 1$, we have $S_t^{b-1,b} = L_{t,b} = F_{t,b}$, and therefore

$$\boldsymbol{\phi_b} = \frac{v_t^{b-1,b}}{\sqrt{\delta_{t,b-1}^{b,b}}} \tag{622}$$

We move to next co-terminal swaption maturing at T_{b-2} for $k = b - 2$. The (620) gives

$$\begin{aligned} &\underbrace{F_{t,b-1}^2 \delta_{t,b-2}^{b-1,b-1}}_{A} \boldsymbol{\phi_{b-1}^2} + \underbrace{2F_{t,b-1} F_{t,b} \phi_b \delta_{t,b-2}^{b,b-1}}_{B} \boldsymbol{\phi_{b-1}} \\ &+ \underbrace{F_{t,b}^2 \phi_b^2 \delta_{t,b-2}^{b,b} - \left(v_t^{b-2,b} S_t^{b-2,b}\right)^2}_{C} = 0 \end{aligned} \tag{623}$$

Assuming that in previous step, the last co-terminal swaption has been calibrated exactly by setting ϕ_b as in (622), an exact calibration to this co-terminal swaption can be achieved by setting ϕ_{b-1} equal to the higher positive solution to the quadratic equation.

By following the above steps, we can derive all ϕ_k for $k = b, \cdots, 2$ recursively and analytically through an exact calibration to a co-terminal swaption maturing at T_{k-1} given that all ϕ_i, $i = b, \cdots, k + 1$ have been previously identified.

Our next step is to fit the model to the upper triangular portion of the volatility matrix. We end up with solving the following optimization problem

$$\begin{gathered} \underset{\alpha,\beta}{\operatorname{Argmin}} \left(\sum_{1 \le i < j = b} w_{i,j}^{swpt} \left(\varsigma_{i,j}^{mkt} - \varsigma_{i,j}^{mdl} \right)^2 \right) \\ \text{subject to } \alpha \in \mathbb{C}_\alpha, \ \beta \in \mathbb{C}_\beta \end{gathered} \tag{624}$$

It should be noted that this calibration has a potential issue. When the initial guess of α is not close to the true value, the computed ϕ vector is unbounded and may be far away from 1 due to parameter redundancy within α and ϕ. To mitigate this issue, we may rescale the ϕ vector by its mean (or geometric mean) in each iteration of the optimization. Experiments show that although the rescaling may lead to an inexact calibration to co-terminal swaptions, the scaling factor will converge and eventually become quite close to 1, and therefore introduce little impact.

11.8. Monte Carlo Simulation

11.8.1. Pricing Vanilla Swaptions

In previous section, we have introduced two analytical approximation formulas for swaption volatility estimation. We may also price the swaptions by means of the MC simulations in LMM. At first we have the payoff function of a payer swaption upon maturity at T_a

$$V_a^{a,b} = \left(S_a^{a,b} - K\right)^+ A_a^{a,b} \quad \text{and} \quad A_a^{a,b} = \sum_{i=a+1}^{b} \tau_i P_{a,i} = \sum_{i=a+1}^{b} \frac{\tau_i}{\prod_{j=a+1}^{i}\left(1+\tau_j L_{a,j}\right)} \tag{625}$$

The present value of the swaption at t can be expressed in various forms under different probability measures, for example

1. Under spot measure $\mathbb{Q}^\eta$

$$V_t^{a,b} = \mathcal{M}_t \widetilde{\mathbb{E}}_t\left[\frac{V_a^{a,b}}{\mathcal{M}_a}\right] = P_{t,\eta}\mathbb{E}_t^\eta\left[\frac{V_a^{a,b}}{\prod_{i=\eta+1}^{a}\left(1+\tau_i L_{i-1,i}\right)}\right] \tag{626}$$

2. Under T_a-forward measure $\mathbb{Q}^a$

$$V_t^{a,b} = P_{t,a}\mathbb{E}_t^a\left[\frac{V_a^{a,b}}{P_{a,a}}\right] = P_{t,a}\mathbb{E}_t^a\left[V_a^{a,b}\right] \tag{627}$$

3. Under T_b-forward measure $\mathbb{Q}^b$ (a.k.a. terminal measure)

$$V_t^{a,b} = P_{t,b}\mathbb{E}_t^b\left[\frac{V_a^{a,b}}{P_{a,b}}\right] = P_{t,b}\mathbb{E}_t^b\left[\left(S_a^{a,b}-K\right)^+ \sum_{i=a+1}^{b}\frac{\tau_i P_{a,i}}{P_{a,b}}\right] = P_{t,b}\mathbb{E}_t^b\left[\mathcal{V}_a^{a,b}\right] \tag{628}$$

Where,

$$\mathcal{V}_a^{a,b} = \left(S_a^{a,b}-K\right)^+ \sum_{i=a+1}^{b}\frac{\tau_i}{P_{a,i,b}} = \left(S_a^{a,b}-K\right)^+ \sum_{i=a+1}^{b}\tau_i \prod_{j=i+1}^{b}\left(1+\tau_j L_{a,j}\right)$$

can be thought of as a sum of cashflows inflated to T_b.

These formulas are mutually equivalent and must produce the same price if we perform MC simulations under respective measures accordingly. However, the formula under the spot measure implies that one must simulate simultaneously the full term structure of the forward rates from $T_{\eta+1}$ up to T_b, mainly due to the stochastic discount factor within the expectation. This is in fact unnecessary if we work in the other two cases, where we only need to simulate the rate dynamics for $L_i,\ i = a+1, \cdots, b$. This is why MC simulation of LMM in most cases is in favor of T-forward measures.

Let us consider a MC simulation for a payer swaption $PS_t^{a,b}$ under the terminal measure $\mathbb{Q}^b$. The swaption price is determined by the rate dynamics of $L_i,\ i = a+1, \cdots, b$ under $\mathbb{Q}^b$, which is given by (581)

$$dL_{t,i} = L_{t,i}\sigma_{t,i}dW_t^b - L_{t,i}\sigma_{t,i}\sum_{j=i+1}^{b}\frac{\tau_j L_{t,j}\rho_{ij}\sigma_{t,j}}{1+\tau_j L_{t,j}}dt, \qquad \forall\ i = a+1, \cdots, b \ \ and \ \ t \le T_{i-1} \tag{629}$$

The above SDE is often discretized in logarithmic form to reduce the numerical instability,

$$\log L_{t+\Delta t,i} = \log L_{t,i} - \sigma_{t,i}\sum_{j=i+1}^{b}\frac{\tau_j L_{t,j}\rho_{ij}\sigma_{t,j}}{1+\tau_j L_{t,j}}\Delta t - \frac{\sigma_{t,i}^2}{2}\Delta t + \sigma_{t,i}\sqrt{\Delta t}\mathcal{N}_i(0,\rho) \tag{630}$$

where $\mathcal{N}_i(0,\rho)$ is the i-th component of a multivariate normal random variable. Apparently, this is not a Markovian process as the drift term is path-dependent. To simulate a realization, we evolve the forward Libor rates $L_i,\ i = a+1, \cdots, b$ simultaneously from present time $t = T_0$ to the swaption maturity T_a. The realized forward rates at T_a are then used to calculate the swap rate $S_a^{a,b}$ and the inflation factors $\prod_{j=i+1}^{b}(1+\tau_j L_{a,j})$, and eventually the payoff $\mathcal{V}_a^{a,b}$. This simulation is repeated m times. The swaption price can then be calculated as an average of the m realized payoffs discounted by $P_{t,b}$.

Notice that in the above formula, the rates and volatilities are actually time dependent, we cannot assume them to be constant within a time step if the time step is large. There are many methods to mitigate this issue. For example, we may use *predictor-corrector* method to minimize the error due to time dependent drift term. The method estimates the drift term using rates $L_{t,j}$ in the first attempt, then use this drift to evolve the rates to get $\tilde{L}_{t+\Delta t,i}$. In the second attempt, the drift is estimated again based on the evolved rates $\tilde{L}_{t+\Delta t,i}$ from the first attempt. The average of the two drift terms is then used to evolve the rates $L_{t+\Delta t,i}$ for current time step.

To minimize the error due to the time dependent volatility term, we may use the mid value $\frac{1}{2}(\sigma_{t,i}+\sigma_{t+\Delta t,i})$ to replace the volatility $\sigma_{t,i}$ in the simulation. If a more accurate approximation is desired, one can use the following formula to run the simulation

$$\log L_{t+\Delta t,i} = \log L_{t,i} - \sum_{j=i+1}^{b} \frac{\tau_j L_{t,j} \Sigma_{t,\Delta t}^{i,j}}{1+\tau_j L_{t,j}} - \frac{\Sigma_{t,\Delta t}^{i,i}}{2} + \mathcal{N}_i(0,\Sigma_{t,\Delta t})$$
$$\text{where,} \qquad \Sigma_{t,\Delta t}^{i,j} = \int_t^{t+\Delta t} \rho_{ij}\sigma_{u,i}\sigma_{u,j}\,du \tag{631}$$

This can be combined with the predictor-corrector method to further improve the simulation accuracy.

11.8.2. Bermudan Swaption by Least Square Monte Carlo

$$t=T_0 \longrightarrow \cdots \longrightarrow \boldsymbol{T_a} \longrightarrow T_{a+1} \longrightarrow \cdots \longrightarrow T_{b-2} \longrightarrow \boldsymbol{T_{b-1}} \longrightarrow \boldsymbol{T_b}$$

Let us define a Bermudan swaption. Suppose at present time $t=T_0$, there is a Bermudan payer swaption with a tenor structure $\{T_a,\cdots,T_b\}$. The option-holder has the right to enter a swap by exercising this swaption at any of the dates $\{T_a,\cdots,T_{b-1}\}$, given that the swaption has not been exercised previously. Upon exercise, the holder immediately enters into a payer swap that matures at T_b.

Since Bermudan swaption has embedded path-dependent feature, we must work out the rate realizations backwards to identify the optimality of exercise at different times. Traditional numerical methods, like the finite difference techniques or binomial trees, are generally unsuited to handle higher-dimensional problems, like the pricing of a Bermudan swaption in LMM, because their computation time grows too quickly as the dimension of the problem increases. Monte Carlo methods are very well suited for higher dimensional problems and path dependency, but have serious problems with early exercise features. Longstaff and Schwartz proposed a promising new algorithm, known as the Least Squares Monte Carlo (LSM) algorithm, for pricing early exercise products by Monte Carlo simulation. The key idea behind the algorithm is to approximate the conditional expected payoff from continuation with least squares approximation down to a set of basic functions.

Here we summarize the Longstaff-Schwartz method in brief as follows

1. The Monte Carlo simulation is performed to generate p paths of the forward rates $L_{i,j}$ for $i = a,\cdots,b-1$ and $j = i+1,\cdots,b$ under the terminal measure $\mathbb{Q}^b$. The rates must evolve from present time $t = T_0$ to the last exercise date T_{b-1}. However, as the t elapses beyond T_a, the length of the underlying swap shrinks, hence the number of the forward rates to be simulated decreases.
2. The rate paths are then processed backwards, starting from the final exercise date T_{b-1}. For $i = b-1$, we calculate the payoff value $\mathcal{V}_{b-1}^{b-1,b}$ for each path using the rate $L_{b-1,b}$. This value can be regarded as the swaption value at T_{b-1} (inflated to T_b) for one path. We define $\mathcal{C}_i$ the swaption value from continuation, then $\mathcal{C}_{b-1} = \mathcal{V}_{b-1}^{b-1,b}$ is the swaption value for continuously holding it up to T_{b-1}.
3. The early exercise is then considered backwards for dates T_i , $i = b-2,\cdots,a$. At time T_i, we calculate the payoff $\mathcal{V}_i^{i,b}$ for each path using the rates $L_{i,j}$, $j = i+1,\cdots,b$. This is the swaption value if being exercised immediately. The option-holder optimally compares the payoff $\mathcal{V}_i^{i,b}$ from immediate exercise with the conditional expected payoff $\mathbb{E}_i^b[\mathcal{C}_{i+1}]$ from continuation and set $\mathcal{C}_i = \mathrm{Max}\left\{\mathcal{V}_i^{i,b} , \mathbb{E}_i^b[\mathcal{C}_{i+1}]\right\}$. The process then moves to T_{i-1} and repeats, until completes at T_a. Now we have $\mathcal{C}_a$, the Bermudan swaption value for one simulated path. (The method to estimate $\mathbb{E}_i^b[\mathcal{C}_{i+1}]$ will be discussed in more details shortly.)
4. We then have p values of $\mathcal{C}_a$, one for each path. The present value of the swaption at $t = T_0$ equals to the average of the $\mathcal{C}_a$ discounted by $P_{0,b}$.

Longstaff and Schwartz suggest to use a simple linear regression to estimate the conditional expected payoff from continuation $\mathbb{E}_i^b[\mathcal{C}_{i+1}]$ at time T_i. It can be estimated through a multivariate linear function

$$\mathbb{E}_i^b[\mathcal{C}_{i+1}] = \hat{\alpha}_i + \hat{\beta}_i' x_i \tag{632}$$

where x_i is a set of $\mathcal{F}_i$-*measurable* basis functions of the relevant state variables. The parameters $\hat{\alpha}_i$ and $\hat{\beta}_i$ are estimated using the cross-sectional information in the simulated paths by regressing the subsequent value of continuation $\mathcal{C}_{i+1}$ on the x_i as in the linear model

$$\mathcal{C}_{i+1} = \alpha_i + \beta_i' x_i + \epsilon_i \tag{633}$$

Usually only the paths with in-the-money $\mathcal{V}_i^{i,b}$ of *immediate exercise* are included in the regression. This is intuitive because out-of-the-money paths give the holder no choice but to keep holding it. As basis functions, we use simple polynomials of the forward rates $L_{i,j}$, $j = i + 2, \cdots, b$, for example, $L_{i,j}$ and $L_{i,j}^2$. The inclusion of swap rate $S_i^{i+1,b}$ may not be critical, because $S_i^{i+1,b}$ is just a function of the forward rates. Higher degree polynomials can make the regression unstable and is not recommended if the polynomials don't possess some orthogonal characteristics (e.g., Legendre polynomials). The fitted value of this regression is an efficient unbiased estimate of the conditional expectation function and allows us to accurately estimate the optimal stopping rule for the option.

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