Local Volatility Model with Stochastic Rates

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Kolmogorov Forward & Backward Equation

- Multidimensional SDE

$$dX_t = A(t, X_t)dt + B(t, X_t)dW_t, \quad \rho dt = dW_t dW_t'$$

- $p(t, x|s, \alpha)$ is the transition probability density function having $X_t = x$ at t given $X_s = \alpha$ at s for $s < t$

- Forward PDE (a.k.a *Fokker-Planck equation*)

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{m} \frac{\partial (A_i p)}{\partial x_i} - \frac{1}{2} \sum_{i,j=1}^{m} \frac{\partial^2 (\Sigma_{ij} p)}{\partial x_i \partial x_j} = 0, \quad \Sigma = B \rho B'$$

$$\lim_{t \to s} p(t, x|s, \alpha) = \delta(x - \alpha)$$

- Backward PDE

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{m} A_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{m} \Sigma_{ij} \frac{\partial^2 p}{\partial x_i \partial x_j} = 0$$

$$\lim_{t \to T} p(T, \beta|t, x) = \delta(x - \beta)$$

General Dupire Local Vol – Part 1

- Spot dynamics, volatility η_t takes a general form

$$\frac{dX_t}{X_t} = \mu_t dt + \eta_t d\widetilde{W}_t, \quad \mu_t = r_t - q_t$$

- European call price

$$\mathcal{C}_{t,K} = \mathbb{E}_s\left[D_{s,t}(X_t - K)^+\right] = \mathbb{E}_s\left[D_{s,t}(X_t - K)\Theta(X_t - K)\right]$$

$$\frac{\partial \mathcal{C}_{t,K}}{\partial K} = -\mathbb{E}_s\left[D_{s,t}\Theta(X_t - K)\right]$$

$$\frac{\partial^2 \mathcal{C}_{t,K}}{\partial K^2} = \mathbb{E}_s\left[D_{s,t}\delta(X_t - K)\right] = \mathbb{E}_s\left[D_{s,t}\middle|X_t = K\right]\mathbb{E}_s[\delta(X_t - K)]$$

The last equality comes from Bayes' rule

- Heaviside step function and Dirac delta function

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1/2 & x = 0 \\ 1, & x > 0 \end{cases}, \quad \Theta(x) = \int_{-\infty}^{x} \delta(u)du, \quad \delta(x) = \frac{d\Theta(x)}{dx}$$

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \quad \text{subject to} \quad \int_{-\infty}^{\infty} \delta(u)du = 1$$

Analogous to the CDF and PDF of a normal distribution with zero mean and infinitesimal variance

General Dupire Local Vol – Part 2 (Ch 3.2)

- Dynamics of payoff function (not a smooth function; applying Ito-Tanaka formula)

$$d\left(D_{s,t}(X_t-K)^+\right) = -r_t D_{s,t}(X_t-K)^+ dt + D_{s,t}\Theta(X_t-K)dX_t + \frac{1}{2}D_{s,t}\delta(X_t-K)(dX_t)^2$$

- Take expectation of both sides

$$\frac{\partial \mathcal{C}_{t,K}}{\partial t} = \mathbb{E}_s\left[\frac{d\left(D_{s,t}(X_t-K)^+\right)}{dt}\right]$$

$$= \mathbb{E}_s\left[D_{s,t}\left(-r_t(X_t-K)\Theta(X_t-K) + \mu_t X_t\Theta(X_t-K) + \frac{1}{2}X_t^2\eta_t^2\delta(X_t-K)\right)\right]$$

$$= \mathbb{E}_s\left[D_{s,t}(r_tK - q_tX_t)\Theta(X_t-K)\right] + \frac{1}{2}\mathbb{E}_s\left[D_{s,t}X_t^2\eta_t^2\delta(X_t-K)\right]$$

$$= K\mathbb{E}_s\left[D_{s,t}r_t\Theta(X_t-K)\right] - \mathbb{E}_s\left[D_{s,t}q_tX_t\Theta(X_t-K)\right] + \frac{1}{2}K^2\mathbb{E}_s\left[D_{s,t}\eta_t^2\middle|X_t=K\right]\mathbb{E}_s[\delta(X_t-K)]$$

- Finally

$$\frac{\mathbb{E}_s\left[D_{s,t}\eta_t^2\middle|X_t=K\right]}{\mathbb{E}_s\left[D_{s,t}\middle|X_t=K\right]} = \frac{\frac{\partial \mathcal{C}_{t,K}}{\partial t} - K\mathbb{E}_s\left[D_{s,t}r_t\Theta(X_t-K)\right] + \mathbb{E}_s\left[D_{s,t}q_tX_t\Theta(X_t-K)\right]}{\frac{1}{2}K^2\frac{\partial^2\mathcal{C}_{t,K}}{\partial K^2}}$$

- If we write the volatility term η_t as a pure local volatility $g(t, X_t)$, which is a deterministic function of X_t

$$g_{t,K}^2 = \frac{\mathbb{E}_s\left[D_{s,t}\eta_t^2\middle|X_t=K\right]}{\mathbb{E}_s\left[D_{s,t}\middle|X_t=K\right]} = \frac{\frac{\partial \mathcal{C}_{t,K}}{\partial t} - K\mathbb{E}_s\left[D_{s,t}r_t\Theta(X_t-K)\right] + \mathbb{E}_s\left[D_{s,t}q_tX_t\Theta(X_t-K)\right]}{\frac{1}{2}K^2\frac{\partial^2\mathcal{C}_{t,K}}{\partial K^2}}$$

Relationship to Classic Dupire Local Vol (Ch 4.2 & 4.3)

- Assuming deterministic rates $\bar{r}_t$ and $\bar{q}_t$, we get the Classic Dupire local vol

$$\xi_{t,K}^2 = \mathbb{E}_s[\eta_t^2 | X_t = K] = \frac{\frac{\partial \mathcal{C}_{t,K}}{\partial t} + \bar{\mu}_t K \frac{\partial \mathcal{C}_{t,K}}{\partial K} + \bar{q}_t \mathcal{C}_{t,K}}{\frac{1}{2} K^2 \frac{\partial^2 \mathcal{C}_{t,K}}{\partial K^2}}$$

  Dupire local variance = Conditional expectation of the (stochastic) variance

- Difference $\beta_{t,K}$ between the general Dupire local volatility $g(t, X_t)$ and the classic Dupire local volatility $\xi(t, K)$

$$\beta_{t,K} = g_{t,K}^2 - \xi_{t,K}^2 = \frac{\mathbb{E}_s\left[D_{s,t}(q_t - \bar{q}_t) X_t \Theta(X_t - K)\right] - K \mathbb{E}_s\left[D_{s,t}(r_t - \bar{r}_t)\Theta(X_t - K)\right]}{\frac{1}{2} K^2 \frac{\partial^2 \mathcal{C}_{t,K}}{\partial K^2}}$$

  the spread $\beta_{t,K}$ becomes zero if the rates r_t and q_t are deterministic

- The general Dupire local vol is constructed on a time and strike 2D grid, which can be calibrated by estimating the two expectations along with the readily available classic Dupire local vols
- Next, we assume Hull-White model for the stochastic rates and general Dupire local vol for the spot process, and attempt to estimate the two expectations numerically

Local Vol Model with Stochastic Rates (Ch 2)

- 3-Factor model: Hull White model for rates, general local vol for spot

$$r_t = f_{s,t} + \int_s^t b_{u,t}\beta_{u,t}du + x_t$$

$$x_t = \int_s^t \beta_{u,t}dW_{1,u}\,, \qquad dx_t = -\lambda_t x_t dt + \sigma_t dW_{1,t}$$

$$\hat{r}_t = \hat{f}_{s,t} + \int_s^t \hat{b}_{u,t}\hat{\beta}_{u,t}du + \hat{x}_t$$

$$\hat{x}_t = -\int_s^t \rho_u^{23}\hat{\beta}_{u,t}g_{u,y}du + \int_s^t \hat{\beta}_{u,t}dW_{2,u}\,, \qquad d\hat{x}_t = -\left(\rho_t^{23}\hat{\sigma}_t g_{t,y} + \hat{\lambda}_t\hat{x}_t\right)dt + \hat{\sigma}_t dW_{2,t}$$

$$dy_t = \left(r_t - \hat{r}_t - \frac{1}{2}g_{t,y}^2\right)dt + g_{t,y}dW_{3,t}, \qquad dW_{i,t}dW_{j,t} = \rho_t^{ij}dt$$

- The x_t, $\hat{x}_t$ and y_t are the state variables
- Define discounted transition probability density function

$$h(t,x,\hat{x},y|s,x_s,\hat{x}_s,y_s) = D_{s,t}p(t,x,\hat{x},y|s,x_s,\hat{x}_s,y_s), \qquad D_{s,t} = \exp\left(-\int_s^t r_u du\right)$$

$D_{s,t}$ is not a function of any of the state variables

- Next, we briefly review the Hull-White model and then derive the forward PDE, i.e. *Fokker-Planck* equation, for the model

1-Factor Hull-White Model (Ch 2)

- 1-Factor Hull White model for both domestic and foreign rates

$$r_t = f_{s,t} + \int_s^t b_{u,t}\beta_{u,t}du + x_t, \quad x_t = \int_s^t \beta_{u,t}dW_u\,, \quad dx_t = -\lambda x_t dt + \sigma_t dW_t$$

$$\beta_{u,t} = e^{-\lambda(t-u)}\sigma_u, \quad b_{u,t} = \int_u^t \beta_{u,v}dv = \frac{1-e^{-\lambda(t-u)}}{\lambda}\sigma_u$$

$\beta_{u,t}$ is the volatility of the instantaneous forward rate $f_{u,t}$

$b_{u,t}$ is the volatility of the zero coupon bond $P_{u,t}$

- Writing the model in the most familiar form, it will be

$$dr_t = \lambda(\theta_t - r_t)dt + \sigma_t dW_t, \quad \theta_t = f_{s,t} + \frac{1}{\lambda}\frac{\partial f_{s,t}}{\partial t} + \frac{1}{\lambda}\int_s^t \beta_{u,t}^2 du$$

In practical applications, it usually takes a time-invariant mean reversion rate λ along with a (deterministic) piecewise constant short rate volatility σ_t

- The model can calibrate to caplets or to co-terminal swaptions (via Jamshidian decomposition) knowing that a forward starting zero coupon bond under T-forward measure is a lognormal martingale (usually fix λ while calibrating σ_t)

$$P_{t,T,V} = \frac{P_{t,V}}{P_{t,T}} = \frac{P_{s,V}}{P_{s,T}}\exp\left(-\int_s^t \frac{b_{u,V}^2 - b_{u,T}^2}{2}du - \int_s^t \left(b_{u,V} - b_{u,T}\right)dW_u\right)$$

$$= \frac{P_{s,V}}{P_{s,T}}\exp\left(-\int_s^t \frac{\left(b_{u,V} - b_{u,T}\right)^2}{2}du - \int_s^t \left(b_{u,V} - b_{u,T}\right)dW_u^T\right), \quad dW_u^T = dW_u + b_{u,T}du$$

Calibration by Forward PDE (Ch 3.1 & Ch 4.1.2)

- Forward PDE

$$\frac{\partial h}{\partial t} = -rh + \frac{\partial(\lambda_t x_t h)}{\partial x} + \frac{\partial\left(\left(\rho_t^{23}\hat{\sigma}_t g_{t,y} + \hat{\lambda}_t \hat{x}_t\right)h\right)}{\partial \hat{x}} - \frac{\partial\left(\left(r_t - \hat{r}_t - \frac{1}{2}g_{t,y}^2\right)h\right)}{\partial y} + \frac{1}{2}\frac{\partial^2(\sigma_t^2 h)}{\partial x^2} + \frac{1}{2}\frac{\partial^2(\hat{\sigma}_t^2 h)}{\partial \hat{x}^2}$$
$$+ \frac{1}{2}\frac{\partial^2(g_{t,y}^2 h)}{\partial y^2} + \frac{\partial^2(\rho_t^{12}\sigma_t\hat{\sigma}_t h)}{\partial x \partial \hat{x}} + \frac{\partial^2(\rho_t^{13}\sigma_t g_{t,y} h)}{\partial x \partial y} + \frac{\partial^2(\rho_t^{23}\hat{\sigma}_t g_{t,y} h)}{\partial \hat{x} \partial y}$$

$$\lim_{t \to s} h(t, x, \hat{x}, y | s, x_s, \hat{x}_s, y_s) = \delta(x - x_s, \hat{x} - \hat{x}_s, y - y_s)$$

- The 3D PDE can be solved by Alternating Direction Implicit (ADI) method
- Steps of calibrating general Dupire local vol $g(t, y)$
  - Start from t_0, using $g(t_0, y) = \xi(t_0, y)$ and 3D Dirac delta function for $h(t_0)$
  - For each time step from t_i to t_{i+1}, use forward PDE to evolve the density from $h(t_i)$ to $h(t_{i+1})$, using the previously calculated $g(t_i, y)$
  - Use the resulting $h(t_{i+1})$ at t_{i+1} to compute the adjustment $\beta(t_{i+1}, y)$, basically evaluate the two expectations in the numerator
  - Use the computed adjustment $\beta(t_{i+1}, y)$ along with the classic Dupire local volatility $\xi(t_{i+1}, y)$ to compute the general local volatility $g(t_{i+1}, y)$ for the next time interval from t_{i+1} to t_{i+2}
  - Repeat steps 2 to 4 until we have evolved the density function all the way to maturity

Pricing by Backward PDE

- The pricing is done through solving the backward PDE using the general Dupire local volatility, along with proper terminal conditions and boundary conditions

$$\frac{\partial V}{\partial t} = rV + \lambda_t x_t \frac{\partial V}{\partial x} + \left(\rho_t^{23}\hat{\sigma}_t g_{t,y} + \hat{\lambda}_t \hat{x}_t\right)\frac{\partial V}{\partial \hat{x}} - \left(r_t - \hat{r}_t - \frac{1}{2}g_{t,y}^2\right)\frac{\partial V}{\partial y} - \frac{\sigma_t^2}{2}\frac{\partial^2 V}{\partial x^2}$$

$$- \frac{\hat{\sigma}_t^2}{2}\frac{\partial^2 V}{\partial \hat{x}^2} - \frac{g_{t,y}^2}{2}\frac{\partial^2 V}{\partial y^2} - \rho_t^{12}\sigma_t\hat{\sigma}_t\frac{\partial^2 V}{\partial x \partial \hat{x}} - \rho_t^{13}\sigma_t g_{t,y}\frac{\partial^2 V}{\partial x \partial y} - \rho_t^{23}\hat{\sigma}_t g_{t,y}\frac{\partial^2 V}{\partial \hat{x}\partial y}$$

- Again, this can be solved by ADI method

Stochastic Local Vol with Stochastic Rates (Ch 5)

- 4 Factor model: assuming stochastic local vol for spot process. The stochastic volatility component is driven by another stochastic process z_t

$$dy_t = \left(r_t - \hat{r}_t - \frac{1}{2}\eta_t^2\right)dt + \eta_t dW_{3,t}, \qquad \eta_t = \gamma_{t,y}\psi_{t,z}$$

$$dz_t = a_{t,z}dt + b_{t,z}dW_{4,t}$$

- Local vol component $\gamma_{t,y}$ can be derived as

$$\frac{\mathbb{E}_s[D_{s,t}\eta_t^2|y_t=k]}{\mathbb{E}_s[D_{s,t}|y_t=k]} = \frac{\mathbb{E}_s[D_{s,t}\gamma_{t,y}^2\psi_{t,z}^2|y_t=k]}{\mathbb{E}_s[D_{s,t}|y_t=k]} = \gamma_{t,k}^2\frac{\mathbb{E}_s[D_{s,t}\psi_{t,z}^2|y_t=k]}{\mathbb{E}_s[D_{s,t}|y_t=k]}$$

$$\Longrightarrow \gamma_{t,k}^2 = g_{t,k}^2\frac{\mathbb{E}_s[D_{s,t}|y_t=k]}{\mathbb{E}_s[D_{s,t}\psi_{t,z}^2|y_t=k]} = g_{t,k}^2\frac{\int_\Omega h(t,x,\hat{x},y=k,z|s,x_s,\hat{x}_s,y_s,z_s)d\Omega}{\int_\Omega h(t,x,\hat{x},y=k,z|s,x_s,\hat{x}_s,y_s,z_s)\psi_{t,z}^2 d\Omega}$$

- Calibration can be challenging: PDE method is basically infeasible for 4-Factor model

- In order to calibrate $\gamma_{t,y}$ for each time step, we must estimate 4 expectations using Monte Carlo method
  - Two for $g_{t,k}^2$: $\mathbb{E}_s[D_{s,t}r_t\Theta(X_t - K)]$ and $\mathbb{E}_s[D_{s,t}q_tX_t\Theta(X_t - K)]$
  - Two as above: $\mathbb{E}_s[D_{s,t}|y_t = k]$ and $\mathbb{E}_s[D_{s,t}\psi_{t,z}^2|y_t = k]$

  The first two expectation is relatively straight forward to calculate. The second two are conditional expectations, which may need special treatment

Reference:

- Deelstra, G. and Rayée, G., *Local Volatility Pricing Models for Long-dated FX Derivatives*, 2012
  https://arxiv.org/pdf/1204.0633